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Quasi-pinning synchronization and stabilization of fractional order BAM neural networks with delays and discontinuous neuron activations

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ABSTRACT

This manuscript concerns quasi-pinning synchronization and $\boldsymbol{\beta}$-exponential pinning stabilization for a class of fractional order BAM neural networks with time-varying delays and discontinuous neuron activations (FBAMNNDDAs). Firstly, under the framework of Filippov solution and fractional-order differential inclusions analysis for the initial value problem of FBAMNNDDAs is presented. Secondly, two kinds of novel pinning controllers according to pinning control technique are designed. By means of fractional order Lyapunov method and designed pinning control strategy, the sufficient criteria is given first to ensure the quasi-synchronization for the dynamic behavior of FBAMNNDDAs. Furthermore, the error bound of pinning synchronization is explicitly evaluated. Thirdly, via Kakutani s fixed point theorem of set-valued map analysis, Razumikhin condition, and a nonlinear pinning controller, the existence and $\boldsymbol{\beta}$-exponential stabilization of FBAMNNDDAs equilibrium point is obtained in the voice of linear matrix inequality (LMI) technique. Fourthly, based on as well as Mittag-Leffler function and growth condition, the global existence of a solution in the Filippov sense of such system is guaranteed with detailed proof. At last, a numerical example with computer simulations are performed to illustrate the effectiveness of proposed theoretical consequences.

1. Introduction

In 1695, the idea of fractional order calculus becomes first off mentioned through German mathematician Leibniz and it failed to attract more attention for a long time since it lack of application background and the complexity. In past few decades, fractional order calculus has superior characteristics over traditional calculus [1,2], and some excellent results on fractional order systems based on fractional-order calculation have been demonstrated, see [3–6]. Within the field of electronics, the version of fractional capacitor, formally called the fractance, has been offered, which describes the fractional differentiation constitutive relationship $\mathcal{I}_t = \mathcal{C}D^{\beta}\mathcal{V}_t$ between $\mathcal{V}_t$ and $\mathcal{I}_t$ passing through it, where $\mathcal{C}$ is the capacitance of the capacitor, $\mathcal{V}_t$ is input voltage, $\mathcal{I}_t$ is current and the fractional order β is identified with the misfortunes of the capacitor. The integer-order capacitor (inductor) is in reality not existing, that's only an approximation of a fractional order capacitor (or inductor) $\mathcal{C}$. The primary reason is that dielectric materials represent capacitor (inductor) pondered the fractional order traits. Consequently, the fractional-order differential equation can accurately describe with a capacitance (or inductor) circuit system [7,8]. Neural networks have found a wide scope of applications in automatic control, combinatorial optimization, image processing and signal processing [9–12]. An electronic implementation of an artificial neural network model, many of the researchers attempted to update the normal capacitor by fractional capacitor, then it creates the fractional order neural network models. Until newly, it has got increasing interests of many researchers and it plays a vital role in synchronization [13,14], state estimation [15], dissipativity [16], passivity [17], stability [18] and stabilization [19] of fractional order neural networks and the research of the fractional order dynamical system has been a hot spot.

As a type of recurrent neural networks, BAM neural networks was firstly predicted by Kosko in 1987. As we recognize, a BAM type of neural network model is a nonlinear feedback network

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model and it contains two sorts of layers like K-layer and L-layer. These layers are always fully interconnected with each layer. Over the beyond few years, the research topic of BAM type neural networks(BAMNNs) dynamics has received great attention because it has successfully applied in many areas, like power systems, mechanics of structures and materials, hetero and auto-associative memories and other related research areas. In practical neural dynamic systems, the arising of time delay is unavoidable due to the limited speed of information transmission. In accordance, there are different kinds of time delays, such as neutral time-delay, constant time-delay, time-varying delay, finite(or infinite) distributed delay, leakage delay and so forth. In fact, the advent of these delays might also have an effect on the deteriorated overall performance of the dynamic of neural network systems. Hence, the study of fractional order delayed BAM neural networks has been broad interests of many scholars, see Refs. [20–23]. It is well known that the neuron activation functions, that's a fundamental relation between the input layer and output layer of a single neuron, plays an essential role in the stabilization, monostability, multistability [24], synchronization and dissipativity analysis of systems. Very recently, the problem of state estimator design of FNNs was established in [15], by using fractional order Lyapunov direct methods. In [17], by using finite-time stability theory and passivity theory, the author investigated about the finite time analysis of fractional order neural networks in the voice of linear matrix inequality. In [25], by using Mittag-Leffler function, Gronwall's inequality and Lyapunov functionals, the author investigated about the existence and Mittag-Leffler stability analysis of fractional order neural networks with time delays. It should be mentioned here, the above mentioned FNNs activations are assumed to be Lipschitz continuous or bounded. But, these conditions are restricted in this paper and not required to monotonicity of the activation functions.

To the best of our author's knowledge, most practical systems are unstable in nature. In this situation, we have applied to some suitable controllers in the practical FNNs system to ensure the corresponding asymptotic behavior and enhance the system performance. During the past few years, various kinds of control techniques have been developed, for instance, sliding mode control [26], state feedback control [27], non-fragile control [28], adaptive control [29], output feedback control [30] and intermittent control [31] so forth. It is noticed that all the aforementioned controllers are applied to every neuron of FNNs, which could be very high priced and impractically. Different from those control techniques, pinning control is more effective because it has been applied to one neuron or the huge number of neurons instead of all neurons. The basic model of classical pinning control strategy is defined as below:

$$M_i(t) = \begin{cases} \tilde{M}_i(t), & \text{if } i \in \{1, 2, \ldots, s\} \\ 0, & \text{if } i \in \{s+1, s+2, \ldots, n\}, \end{cases}$$

where n denotes the number of neurons in FNNs, s denotes the number of directly controlled neurons, $\tilde{M}_i(t)$ denotes the normal control inputs, which is added to each neurons and $M_i(t)$ is appropriate pinning control inputs. Obviously, $n-s$ neurons are not directly controlled.

As a collective behavior of a discontinuous FNNs dynamical system, stabilization and synchronization concepts are very important owing to its wide application in both control theory and system identification respectively. Recently, some remarkable results have been well addressed on synchronization and stabilization of FNNs models in the overview of the earlier literature, kindly see Refs. [19,30,32]. It is noticed in the previous mentioned FNNs results, the neuron activation functions are assumed to be common Lipschitz continuous and their tools are not suitable for the discontinuous FNNs dynamical system. However, some excellent results relevant to stability and synchronization of FNNs with discontinuous activation has been paid in the present assessment of literature, see Refs. [33–37]. For instance in [33], the authors gave some asymptotical synchronization criteria for time varying delayed FNNs with discontinuous activation by means of state feedback and adaptive feedback control. In [37], by using some inequality scaling skills, Lyapunov direct method and linear feedback controller including discontinuous term the author investigated about the finite time stabilization of FNNs with discontinuous activation and uncertain parameter.

Due to the merits the of pinning control method, till present, few results on integer order stability and synchronization of neural networks with continuous or discontinuous activations results have been concerned. For example, Dongshu et al. [38] investigated the pinning control policy of synchronization criteria for integer order discontinuous Cohen-Grossberg neural networks with mixed time-delays, while Dongshu et al. [39] studied the robust synchronization analysis of integer order discontinuous Cohen-Grossberg neural networks with time varying delays. Global exponential pinning stabilization of neural networks with delays was focused in [40]. Besides, the exponential pinning impulsive stabilization of integer order mixed time delayed reaction-diffusion neural networks were analyzed in [41]. Nevertheless, there are null results in dynamical behaviors of stability and synchronization of neural networks with fractional order derivative and discontinuous neuron activations and is it become still an open problem.

Motivation by above discussion and inspiration ideas of Refs. [42–46], we try to analyze the quasi-synchronization and β-exponential stabilization for time-varying delayed FBAMNNs with discontinuous activations. The crucial novelty of this research work is highlighted in the following aspects.

1. Firstly, two novel pinning controllers are developed to ensure the quasi-synchronization and β-exponential stabilization results.
2. Secondly, based on developed control policy and suitable Lyapunov functional, a list of quasi-synchronization analysis is introduced for time-varying delayed FBAMNNS with discontinuous neuron activations. Moreover, the proposed Lyapunov functional is also dependent on the definition of fractional integrals of order $0 < \beta < 1$.
3. Thirdly, the proper designing algorithm of pinning control policy is proposed.
4. Fourthly, via Kakutaniâs fixed point theorem, functional differential inclusion analysis and the framework of Filippov solutions, the existence of equilibrium point is presented, and new β-exponential stabilization for FBAMNNS with discontinuous neuron activations is discussed in the voice of LMI approach.
5. Finally, one numerical examples with computer simulations is presented to illustrate the feasibility of obtained theoretical results.

Nomenclature. In this proposal, $\mathbb{R}$ represents the space of real numbers, $\mathbb{R}^m$ represents the space of m-D Euclidean space, respectively, and $\mathbb{R}^{m\times m}$ stands for a set of all $m \times m$ real matrices. Let $v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m$ denotes a column vector, where superscript T stands for the transpose operator. Given a symmetric matrix G, $G > 0(G \geq 0))$ means positive-definite (positive semi-definite), that is $v^T D v > 0(\geq 0)$ for any $0 \neq z \in \mathbb{R}^m$, $\lambda^*_{\min}(G)(\lambda^*_{\max}(G))$ denote the minimal(maximal) Eigenvalues of real matrices D, respectively. The two norm of vector v is defined by $\|v\|_2 = \sqrt{v_1^2 + \ldots + v_m^2}$.

2. Preliminaries and system description

In this subsection, we will first recall some basic definitions and important properties in fractional order calculus.

Definition 2.1. [47] The Riemann-Liouville fractional integral order $\beta \in (0, 1)$ for a function $k(t)$ is defined as

$$\mathcal{I}^{\beta} k(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\omega)^{\beta-1} k(\omega)\, \mathrm{d}\omega,$$

where $\Gamma(\cdot)$ is Euler's Gamma function.

Definition 2.2. [47] The Caputo fractional-order derivative with order β for a differential function $h(t)$ is defined as

$$D^{\beta} h(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{h^{(m)}(\omega)}{(t-\omega)^{\omega-m+1}}\, \mathrm{d}\omega,$$

where $t \geq 0$ and $m-1 < \beta < m \in \mathbb{Z}^+$. Peculiarly, when $\beta \in (0, 1)$,

$$D^{\beta} h(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{h'(\omega)}{(t-\omega)^{\beta}}\, \mathrm{d}\omega.$$

Furthermore, the following Caputo fractional-order derivative properties are necessary to derivative our main results.

Property 1. *For $m-1 < \beta < m$, we have*

$$\mathcal{I}^{\beta}\left[D^{\beta} k(t)\right] = k(t) - \sum_{j=0}^{m-1} \frac{(t-t_0)^j}{j!} k^{(j)}(t_0),\ \ \beta \geq 0.$$

Especially, $0 < \beta < 1$, *one has*

$$\mathcal{I}^{\beta}\left[D^{\beta} k(t)\right] = k(t) - k(0).$$

Property 2. *For any arbitrary constants f_1 and f_2, the linear property of Caputo derivative is denoted by:*

$$D^{\beta}\left[f_1 k_1(t) + f_2 k_2(t)\right] = f_1 D^{\beta} k_1(t) + f_2 D^{\beta} k_2(t).$$

Definition 2.3. [47] The two parameters Mittag-Leffler function with $\beta > 0, \bar{\beta} > 0$ has expressed in the following form:

$$\mathcal{E}_{\beta,\bar{\beta}}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\beta j + \bar{\beta})},$$

where $z \in \mathbb{C}$. For $\bar{\beta} = 1$, its one parameter function Mittag-Leffler is described as

$$\mathcal{E}_{\beta}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\beta j + 1)} = \mathcal{E}_{\beta,1}(z).$$

Particularly, $\mathcal{E}_{1,1}(z) = \exp\{z\}$, when $\beta = \bar{\beta} = 1$.

In this manuscript, we consider the drive system of fractional order BAM neural networks (FBAMNNs) model:

$$\begin{cases} D^{\beta} k_i(t) = -a_i k_i(t) + \sum_{j=1}^{m} v_{ij} h_j(l_j(t)) + \sum_{j=1}^{m} w_{ij} h_j(l_j(t-\eta(t))) + J_i \\ D^{\beta} l_j(t) = -b_j l_j(t) + \sum_{i=1}^{n} p_{ji} g_i(k_i(t)) + \sum_{i=1}^{n} q_{ji} g_i(k_i(t-\eta(t))) + I_j \\ k_i(\omega) = \phi_i(\omega),\ i \in \mathfrak{I}_n = \{1, 2\ldots, n\}, \\ l_j(\omega) = \psi_j(\omega),\ j \in \mathfrak{I}_m = \{1, 2, \ldots, m\},\ \forall\, \omega \in [-\eta, 0], \end{cases} \tag{1}$$

and the vector form is

$$\begin{cases} D^{\beta} k(t) = -Ak(t) + Vh(l(t)) + Wh(l(t-\eta(t))) + J \\ D^{\beta} l(t) = -Bl(t) + Pg(k(t)) + Qg(k(t-\eta(t))) + I \\ k(\omega) = \phi(\omega),\ l(\omega) = \psi(\omega),\ \forall\, \omega \in [-\eta, 0], \end{cases} \tag{2}$$

where $K = \{k_1, \ldots, k_n\}$ and $L = \{l_1, \ldots, l_m\}$ are bi-layers in FBAMNNs model (1), D^{β} is the Caputo fractional order β lies between 0 and 1, $k(t) = \big(k_1(t), \ldots, k_n(t)\big)^T$, $l(t) = \big(l_1(t), \ldots, l_m(t)\big)^T$ represents the vector of neuron states at time t in K-layer and L-layer, respectively; $I = (I_1, \ldots, I_m)^T$ and $J = (J_1, \ldots, J_n)^T$ denotes the ith and jth components of constant external input vectors; $A = diag\{a_1, \ldots, a_n\} > 0$, $B = diag\{b_1, \ldots, b_m\} > 0$ are self feedback inhibitions in K-layer and L-layer, respectively, where $a_i > 0, b_j > 0$ for $i \in \mathfrak{I}_n$, $j \in \mathfrak{I}_m$, respectively; $V = \big(v_{ij}\big)_{n\times m} \in \mathbb{R}^{n\times m}$ and $W = \big(w_{ij}\big)_{n\times m} \in \mathbb{R}^{n\times m}$ stands for the synaptic connection strengths at time t and $t-\eta(t)$ in K-layer, respectively, while $P = \big(p_{ji}\big)_{m\times n} \in \mathbb{R}^{m\times n}$ and $Q = \big(q_{ji}\big)_{m\times n} \in \mathbb{R}^{m\times n}$ are the similar statuses in L-layer; $h(l(t)) = \big(h_1(l_1(t)), \ldots, h_m(l_m(t))\big)^T$ and $g(k(t)) = \big(g_1(k_1(t)), \ldots, g_n(l_n(t))\big)^T$ are the discontinuous neuron activations of the ith neurons and jth neurons respectively; the time-varying delay $\eta(t)$ is bounded on the interval $[0, +\infty)$ and they satisfy $0 \leq \hat{\eta} \leq \eta(t) \leq \eta$, $0 \leq \eta'(t) \leq \hat{\eta} < 1$ for $t \in [0, +\infty)$.

In this manuscript, the neuron activations $h_j,\ j \in \Im_m$ and $g_i,\ i \in \Im_n$ are considered with the presence of discontinuity. As a result, the classical solution for fractional order differential equations does not applicable to FBAMNNs model (1). In this case, we need to study the concept of Filippov solutions (1) of considering fractional order discontinuous right hand side system [48].

Now, the set-valued map analysis [49,50] of $h(l)$ at $l \in \mathbb{R}^m$ and $g(k)$ at $k \in \mathbb{R}^n$ are defined as below:

Definition 2.4. (Filippov Regularization). Consider the following fractional order differential system:

$$\begin{cases} D^\beta \nu(t) = h(t, \nu),\ t > 0,\ \nu \in \mathbb{R}^m, \\ \nu(0) = \nu_0, \end{cases} \tag{3}$$

where $h(t, \nu)$ is discontinuous in ν. The Fillipov set-valued map $\mathcal{F} : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is defined as:

$$\mathcal{F}(t, \nu) = \bigcap_{\tau>0} \bigcap_{\mu(\mathcal{M})=0} \overline{co}\Big[h\big(t, \mathcal{B}\big(\nu, \tau\big) \backslash \mathcal{M}\big)\Big]$$

where $\mathcal{B}\big(\nu, \tau\big) = \{\tilde{\nu};\ \|\tilde{\nu} - \nu\| \le \tau\}$, $\mathcal{M} \subseteq \mathbb{R}^m$ and $\mu(\mathcal{M})$ is the Lebesgue measure of set $\mathcal{M}$. A vector function $\nu(t)$ defined on $I \subseteq \mathbb{R}$ is called a Filippov solution of system (3), if it is absolutely continuous on any subinterval a non degenerate interval $[t_1, t_2]$ of I, for a.a. $t \in I$, $\nu(t)$ satisfies the differential inclusion: $D^\beta \nu(t) \in \mathcal{F}(t, \nu)$

Definition 2.5. (Filippov solutions). A function $(k^T(t), l^T(t))^T$ is said to be Filippov solution of (1) on $[-\eta, T),\ 0 < T \le +\infty$ if

1. $(k^T(t), l^T(t))^T$ is continuous on $[-\eta, T)$ and absolutely continuous on [0,T).
2. $k(t)$ and $l(t)$ satisfies

$$\begin{cases} D^\beta k_i(t) \in -a_i k_i(t) + \sum_{j=1}^m v_{ij} \overline{co}\{h_j(l_j(t))\} + \sum_{j=1}^m w_{ij} \overline{co}\{h_j(l_j(t - \eta(t)))\} + J_i \\ D^\beta l_j(t) \in -b_j l_j(t) + \sum_{i=1}^n p_{ji} \overline{co}\{g_i(k_i(t))\} + \sum_{i=1}^n q_{ji} \overline{co}\{g_i(k_i(t - \eta(t)))\} + I_j, \end{cases} \tag{4}$$

a.a. $t \in [0, T)$, where

$$\overline{co}\{h_j(l_j)\} = \Big[\min\{h_j(l_j^-), h_j(l_j^+)\},\ \max\{h_j(l_j^-), h_j(l_j^+)\}\Big]$$

$$\overline{co}\{g_i(k_i)\} = \Big[\min\{g_i(k_i^-), g_i(k_i^+)\},\ \max\{g_i(k_i^-), g_i(k_i^+)\}\Big].$$

By virtue of measurable selection theorem, there exist a measureable function $\lambda = \big(\lambda_1, \ldots, \lambda_m\big)^T : [\eta, T) \to \mathbb{R}^m$ and $\mu = \big(\mu_1, \ldots, \mu_n\big)^T : [\eta, T) \to \mathbb{R}^n$ such that $\lambda_j(t) \in \overline{co}\{h_j(l_j(t))\}$, $\mu_i(t) \in \overline{co}\{g_i(k_i(t))\}$ such that

$$\begin{cases} D^\beta k_i(t) = -a_i k_i(t) + \sum_{j=1}^m v_{ij} \lambda_j(t) + \sum_{j=1}^m w_{ij} \lambda_j(t - \eta(t)) + J_i \\ D^\beta l_j(t) = -b_j l_j(t) + \sum_{i=1}^n p_{ji} \mu_i(t) + \sum_{i=1}^n q_{ji} \mu_i(t - \eta(t)) + I_j, \\ k_i(\omega) = \phi_i(\omega),\ \mu_i(\omega) = \tilde{\phi}_i(\omega),\ i \in \Im_n, \\ l_j(\omega) = \psi_j(\omega),\ \lambda_j(\omega) = \tilde{\psi}_j(\omega),\ j \in \Im_m,\ \forall\ \omega \in [-\eta, 0] \end{cases} \tag{5}$$

a.a. $t \in [0, T)$. In this manuscript, we consider the system (1) as drive system, the associated controlled response system is given as follows:

$$\begin{cases} D^\beta \tilde{k}_i(t) = -a_i \tilde{k}_i(t) + \sum_{j=1}^m v_{ij} h_j(\tilde{l}_j(t)) + \sum_{j=1}^m w_{ij} h_j(\tilde{l}_j(t - \eta(t))) + J_i + L_i(t) \\ D^\beta \tilde{l}_j(t) = -b_j \tilde{l}_j(t) + \sum_{i=1}^n p_{ji} g_i(\tilde{k}_i(t)) + \sum_{i=1}^n q_{ji} g_i(\tilde{k}_i(t - \eta(t))) + I_j + N_j(t) \\ \tilde{k}_i(\omega) = \phi_i^*(\omega),\ i \in \Im_n, \\ \tilde{l}_j(\omega) = \psi_j^*(\omega),\ j \in \Im_m,\ \forall\ \omega \in [-\eta, 0] \end{cases} \tag{6}$$

and the vector form is

$$\begin{cases} D^\beta \tilde{k}(t) = -A\tilde{k}(t) + Vh(\tilde{l}(t)) + Wh(\tilde{l}(t - \eta(t))) + J + L(t) \\ D^\beta \tilde{l}(t) = -B\tilde{l}(t) + Pg(\tilde{k}(t)) + Qg(\tilde{k}(t - \eta(t))) + I + N(t) \\ \tilde{k}(\omega) = \phi^*(\omega),\ \tilde{l}(\omega) = \psi^*(\omega),\ \forall\ \omega \in [-\eta, 0], \end{cases} \tag{7}$$

where $\tilde{k}(t) = \big(\tilde{k}_1(t), \ldots, \tilde{k}_n(t)\big)^T$ and $\tilde{l}(t) = \big(\tilde{l}_1(t), \ldots, \tilde{l}_m(t)\big)^T$ are the vector of neuron states at time t of response system, and $L(t) = (L_1(t), \ldots, L_n(t))^T$, $N(t) = \big(N_1(t), \ldots, N_m(t)\big)^T$ are the pinning control input vectors, which is designed in later. The other parameters are similar dynamical meanings as those in drive system (1). And from system (6), we have

$$\begin{cases} D^\beta \tilde{k}_i(t) \in -a_i \tilde{k}_i(t) + \sum_{j=1}^m v_{ij} \overline{co}\{h_j(\tilde{l}_j(t))\} + \sum_{j=1}^m w_{ij} \overline{co}\{h_j(\tilde{l}_j(t - \eta(t)))\} + J_i + L_i(t) \\ D^\beta \tilde{l}_j(t) \in -b_j \tilde{l}_j(t) + \sum_{i=1}^n p_{ji} \overline{co}\{g_i(\tilde{k}_i(t))\} + \sum_{i=1}^n q_{ji} \overline{co}\{g_i(\tilde{k}_i(t - \eta(t)))\} + I_j + N_j(t), \end{cases} \tag{8}$$

a.a. $t \in [0, T)$. Or there exist $\alpha_j(t) \in \overline{co}\{h_j(\tilde{l}_j(t))\}$ and $\beta_i(t) \in \overline{co}\{g_i(\tilde{k}_i(t))\}$ such that

$$\begin{cases} D^\beta \tilde{k}_i(t) = -a_i\tilde{k}_i(t) + \sum_{j=1}^m v_{ij}\alpha_j(t) + \sum_{j=1}^m w_{ij}\alpha_j(t-\eta(t)) + J_i + L_i(t) \\ D^\beta \tilde{l}_j(t) = -b_j\tilde{l}_j(t) + \sum_{i=1}^n p_{ji}\beta_i(t) + \sum_{i=1}^n q_{ji}\beta_i(t-\eta(t)) + I_j + N_j(t), \\ \tilde{k}_i(\omega) = \phi_i^*(\omega),\ \tilde{\beta}_i(\omega) = \tilde{\phi}_i^*(\omega),\ i \in \mathfrak{I}_n, \\ \tilde{l}_j(\omega) = \psi_j^*(\omega),\ \tilde{\phi}_j(\omega) = \tilde{\psi}_j^*(\omega),\ j \in \mathfrak{I}_m,\ \forall\ \omega \in [-\eta, 0]. \end{cases} \tag{9}$$

From response system (9)[or (7)] and drive system (6)[or (1)], the synchronization error dynamical system is described as follows:

$$\begin{cases} D^\beta u_i(t) = -a_i u_i(t) + \sum_{j=1}^m v_{ij}\tilde{\alpha}_j(t) + \sum_{j=1}^m w_{ij}\tilde{\alpha}_j(t-\eta(t)) + L_i(t) \\ D^\beta z_j(t) = -b_j z_j(t) + \sum_{i=1}^n p_{ji}\tilde{\beta}_i(t) + \sum_{i=1}^n q_{ji}\tilde{\beta}_i(t-\eta(t)) + N_j(t), \\ u_i(\omega) = \phi_i^*(\omega) - \phi_i(\omega), i \in \mathfrak{I}_n, \\ z_j(\omega) = \psi_j^*(\omega) - \psi_j(\omega), j \in \mathfrak{I}_m,\ \forall\ \omega \in [-\eta, 0], \end{cases} \tag{10}$$

where $u_i(t) = \tilde{k}_i(t) - k_i(t)$, $z_j(t) = \tilde{l}_j(t) - l_j(t)$, $\tilde{\alpha}_j(t) = \alpha_j(t) - \lambda_j(t) \in \overline{co}\{h_j(\tilde{l}_j(t))\} - \overline{co}\{h_j(l_j(t))\}$ and $\tilde{\beta}_i(t) = \beta_i(t) - \mu_i(t) \in \overline{co}\{g_i(\tilde{k}_i(t))\} - \overline{co}\{g_i(k_i(t))\}$.

For further quasi-synchronization and β-exponential stabilization results, we need the following key definitions, assumptions and related lemma's.

Definition 2.6. A constant vector $\left(k^{*^T}, l^{*^T}\right)^T = \left(k_1^*, \ldots, k_n^*, l_1^*, \ldots, l_m^*\right)^T \in \mathbb{R}^{n+m}$ is said to be an equilibrium point of FBAMNNs in the Filippov' sense if and only if $k^* = \left(k_1^*, \ldots, k_n^*\right)^T \in \mathbb{R}^n$ and $l = \left(l_1^*, \ldots, l_m^*\right)^T \in \mathbb{R}^m$ satisfy the following conditions:

$$\begin{cases} 0 \in -a_i k_i^* + \sum_{j=1}^m v_{ij}\overline{co}\{h_j(l_j^*)\} + \sum_{j=1}^m w_{ij}\overline{co}\{h_j(l_j^*)\} + J_i \\ 0 \in -b_j l_j^* + \sum_{i=1}^n p_{ji}\overline{co}\{g_i(k_i^*)\} + \sum_{i=1}^n q_{ji}\overline{co}\{g_i(k_i^*)\} + I_j. \end{cases}$$

Or equivalently, there exists $\lambda_j^* \in \overline{co}\{h_j(l_j^*)\}$, $\mu_i^* \in \overline{co}\{g_i(k_i^*)\}$, such that

$$\begin{cases} 0 = -a_i k_i^* + \sum_{j=1}^m v_{ij}\lambda_j^* + \sum_{j=1}^m w_{ij}\lambda_j^* + J_i \\ 0 = -b_j l_j^* + \sum_{i=1}^n p_{ji}\mu_i^* + \sum_{i=1}^n q_{ji}\mu_i^* + I_j. \end{cases}$$

Definition 2.7. The FBAMNNs (1) is said to be β-exponential sable to an equilibrium point $\left(k^{*^T}, l^{*^T}\right)^T = \left(k_1^*, \ldots, k_n^*, l_1^*, \ldots, l_m^*\right)^T$, if for any initial conditions $k(\omega) \in \mathcal{C}\big([-\eta, 0], \mathbb{R}^n\big)$ and $l(\omega) \in \mathcal{C}\big([-\eta, 0], \mathbb{R}^m\big)$, there exist a constants $\mathcal{K} > 0$ and $\kappa > 0$ such that

$$\left[\left\|k(t) - k_i^*\right\| + \left\|l(t) - l^*\right\|\right] \le \mathcal{K} \sup_{\omega \in [-\eta, 0]} \left[\|k(\omega) - k^*\| + \|l(\omega) - l^*\|\right] \exp\left\{-\kappa t^\beta\right\},\ t \ge 0.$$

Definition 2.8. The FBAMNNs drive system (1) and controlled response system (7) are said to be quasi-synchronized if for any initial conditions $k_i(\omega),\ \tilde{k}_i(\omega) \in \mathcal{C}\big([-\eta, 0], \mathbb{R}^n\big)$ and $l_j(\omega),\ \tilde{l}_j(\omega) \in \mathcal{C}\big([-\eta, 0], \mathbb{R}^m\big)$ for $i \in \mathfrak{I}_n,\ j \in \mathfrak{I}_m$, there exist a small error bound $\tilde{\mathcal{H}} > 0$ such that

$$\lim_{t \to +\infty}\left[\left(\tilde{k}_i(t) - k_i(t)\right) + \left(\tilde{l}_j(t) - l_j(t)\right)\right] \le \tilde{\mathcal{H}},\ t \ge t_0$$

where t_0 the observation of starting time.

Remark 2.9. In view of Definition 2.8, the systems are said to be asymptotically stable (or asymptotically synchronized) if $\tilde{\mathcal{H}} = 0$.

Assumption $[\mathcal{A}_1]$. For every $i \in \mathfrak{I}_n,\ j \in \mathfrak{I}_m$, suppose the discontinuous activations $g_i, h_j : \mathbb{R} \to \mathbb{R}$ are bounded $\left(|g_i(\cdot)| \le \varpi_i^l,\ |h_j(\cdot)| \le \varpi_j^k\right)$ and continuous function excluding for a finite number of jump discontinuities υ_f on every bounded interval. Furthermore, there exist a left limits $g_i(\upsilon_f^-),\ h_j(\upsilon_f^-)$ and right limits $g_i(\upsilon_f^+),\ h_j(\upsilon_f^+)$, respectively.

Assumption $[\mathcal{A}_2]$. For every $i \in \mathfrak{I}_n,\ j \in \mathfrak{I}_m$, there exist positive constants $\mathcal{R}_i^l,\ \mathcal{R}_j^k,\ \pi_i^l$ and π_j^k such that

$$\left|\alpha_j(t) - \lambda_j(t)\right| \le \mathcal{R}_j^k\left|z_j(t)\right| + \pi_j^k,\ \left|\beta_i(t) - \mu_i(t)\right| \le \mathcal{R}_i^l\left|u_i(t)\right| + \pi_i^l,$$

where $\alpha_j(t) - \lambda_j(t) \in \overline{co}\{h_j(\tilde{l}_j(t))\} - \overline{co}\{h_j(l_j(t))\}$ and $\beta_i(t) - \mu_i(t) \in \overline{co}\{g_i(\tilde{k}_i(t))\} - \overline{co}\{g_i(k_i(t))\}$.

Assumption $[\mathcal{A}_3]$. For every $i \in \mathfrak{I}_n,\ j \in \mathfrak{I}_m$, suppose $\mathcal{F}$ satisfies a growth condition, then there exist positive constants $\tilde{\mathcal{R}}_i^l,\ \tilde{\mathcal{R}}_j^k,\ \tilde{\pi}_i^l$ and $\tilde{\pi}_j^k$ such that

$$\left|\mathcal{F}[h_j(l_j(t))]\right| = \sup_{\varsigma_1 \in \mathcal{F}[h_j(l_j(t))]} |\varsigma_1| \le \tilde{\mathcal{R}}_j^l\left|l_j(t)\right| + \tilde{\pi}_j^l,$$

$$\left|\mathcal{F}[g_i(k_i(t))]\right| = \sup_{\varsigma_2 \in \mathcal{F}[g_i(k_i(t))]} |\varsigma_2| \le \tilde{\mathcal{R}}_i^k\left|k_i(t)\right| + \tilde{\pi}_i^k,$$

where $\mathcal{F}[h_j(l_j(t))] = \overline{co}\{h_j(l_j(t))\}$ and $\mathcal{F}[g_i(k_i(t))] = \overline{co}\{g_i(k_i(t))\}$.

Remark 2.10. When the activations are assumed to be continuous activations, then $\overline{co}\{h_j(l_j)\} = \{h_j(l_j)\}$ and $\overline{co}\{g_i(k_i)\} = \{g_i(k_i)\}$ are singleton. Suppose, there is one discontinuous point exist in Assumption $[\mathcal{A}_2]$, then $\pi_j^k = \pi_i^l = 0$. In this particular case, the neurons continuous activations are satisfied the common Lipschitz-continuous on $\mathbb{R}$. i.e., there exist positive constants $\mathcal{R}_i^l > 0$ and $\mathcal{R}_j^k > 0$ such that

$$\left|h_j(\tilde{l}_j(t)) - h_j(l_j(t))\right| \leq \mathcal{R}_j^k \left|z_j(t)\right|, \ \left|g_i(\tilde{k}_i(t)) - g_i(k_i(t))\right| \leq \mathcal{R}_i^l \left|u_i(t)\right|,$$

for $i \in \Im_n$ and $j \in \Im_m$.

The following Kakutaniâs fixed point theorem is very useful tool to guarantee of the existence of an equilibrium point for the considered model.

Lemma 2.11. *[49] Let E be a compact convex subset of a Banach space X, if the set-valued map* $\Omega : E \rightarrow P_{Kc}(E) = \{E \subseteq X :$ nonempty convex compact set$\}$ *is an upper semi continuous convex compact map* Ω *has fixed point in E. That is* $\kappa \in E$, *such that* $\kappa \in \Omega(\kappa)$.

Lemma 2.12. *[51] A continuous function* $y(t)$ *is defined on the interval* $[0, +\infty)$ *and for* $0 < \beta < 1$, *if there exist two positive constants* $\mathcal{P}_1 > 0$ *and* $\mathcal{P}_2 > 0$ *such that*

$$y(t) \leq -\mathcal{P}_1 D^{-\beta} v(t) + \mathcal{P}_2,$$

then

$$y(t) \leq \mathcal{P}_2 \mathcal{E}_\beta \Big[- \mathcal{P}_1 t^\beta \Big]$$

where $\beta > 0$ *is a positive constant,* $\mathcal{E}_\beta$ *is a Mittag-Leffler one parameter function and* $\Gamma(\cdot)$ *is a Gamma function.*

Lemma 2.13. *[52] For* $0 < \beta < 1$ *and let* $u(t)$ *is the continuous differentiable function, then for any* $t \geq 0$,

$$D^\beta \frac{1}{2} u^2(t) \leq u(t) D^\beta u(t), \ \forall \ \beta \in (0, 1].$$

Lemma 2.14. *[53] For* $\epsilon \geq 1$ *and if* $u_1, \ldots, u_m \geq 0$, *then we have*

$$m^{1-\epsilon} \left[\sum_{j=1}^m u_j \right]^\epsilon \leq \sum_{j=1}^m u_j^\epsilon.$$

Lemma 2.15. *[54] For* $0 < \beta < 1$ *and let* $u(\cdot) : [t_0 - \eta, +\infty) \rightarrow (-\infty, +\infty)$ *be a continuous function such that*

$$D_{0,t}^\beta \, u(t) \leq -\gamma_1 u(t) + \gamma_2 \max_{t-\eta \leq \omega \leq t} u(\omega) \ t \geq 0,$$

then there exist a positive scalars ψ_1, ψ_2 *and* $t^* > t_0 + \eta$ *such that*

$$u(t) \leq \psi_1 \mathcal{E}_\beta \big(- \psi_2 (t - t_0)^\beta \big), \ t \geq t^*,$$

where $\gamma_1 > \gamma_2 > 0$.

Remark 2.16. Let $t \geq t_0$, then monotonic decreasing function $\mathcal{E}_\beta \big(- \psi_2 (t - t_0)^\beta \big)$ satisfy the following condition:

$$\mathcal{E}_\beta \big(- \psi_2 (t - t_0)^\beta \big) \in [0, 1] \ for \ \psi_2 \geq 0.$$

Lemma 2.17. *[55]. For any vectors* $u, \ v \in \mathbb{R}^n$ *and positive definite matrix* Λ, *then*

$$2u^T v \leq u^T \Lambda u + v \Lambda^{-1} v^T.$$

Remark 2.18. When $\beta = 1$, model (1) degenerates into integer order exponential stabilization and pinning synchronization of BAM neural networks with discontinuous activation.

Remark 2.19. Our proposed discontinuous neural networks model can be improved time-varying delay term into constant delay term, the following quasi-synchronization and stabilization results are still true for quasi-synchronization and β-exponential pinning stabilization of fractional order delayed neural networks with discontinuous neuron activation and constant delays.

3. Quasi-synchronization results

In this section, a classical pinning control is designed and quasi-synchronization criteria is discussed for drive-response systems with discontinuous activations.

Novel pinning control is a method which just needs control partial nodes to realize quasi-synchronization for the whole systems. Without loss of generality, we will primary randomly select $\check{s}$ neurons from all neurons in one layer and $\hat{s}$ neurons are randomly chosen in another layer. Referring to the basic principle of pinning control in introduction section, the novel pinning controller is defined as:

$$\begin{cases} M_i(t) = \begin{cases} -\xi sgn\{u_i(t)\} \times \left[\frac{\sum_{i=1}^n |u_i(t)|}{\sum_{i=1}^{\hat{s}} |u_i(t)|} \right] \times \sum_{j=1}^n |u_j(t)| & \text{if } i \in \Im_{\hat{s}} = \{1, 2, \ldots, \hat{s}\} \\ 0 & \text{if } i \in \Im_n \backslash \Im_{\hat{s}} = \{\hat{s} + 1, \ldots, n\} \end{cases} \\ N_j(t) = \begin{cases} -\varepsilon sgn\{z_j(t)\} \times \left[\frac{\sum_{j=1}^m |z_j(t)|}{\sum_{i=1}^{\check{s}} |z_j(t)|} \right] \times \sum_{i=1}^m |z_i(t)| & \text{if } j \in \Im_{\check{s}} = \{1, 2, \ldots, \check{s}\} \\ 0 & \text{if } j \in \Im_m \backslash \Im_{\check{s}} = \{\check{s} + 1, \ldots, m\}, \end{cases} \end{cases} \tag{11}$$

ξ and ε are adjustable positive constants.

In the following, we will deal with the quasi-synchronization of the drive-response (1) and (6) via a novel pinning control approach.

Theorem 3.1. *Assuming the conditions $[\mathcal{A}_1]-[\mathcal{A}_2]$ hold, and then drive-system (1) and controlled response system (6) can be quasi-synchronized via pinning controller (11) if the following algebraic conditions are satisfied:*

$$\mathcal{H} = \min\left\{\min_{i\in\Im_n}\{\xi_{1i}\},\ \min_{j\in\Im_m}\{\varepsilon_{1j}\}\right\} > \mathcal{L} = \max\left\{\max_{i\in\Im_n}\{\xi_{2i}\},\ \max_{j\in\Im_m}\{\varepsilon_{2j}\}\right\} > 0,$$

$$\Phi = \left\{\max_{i\in\Im_n}\left\{\frac{1}{2\varsigma_3}\Phi_{1i}^2\right\} + \max_{j\in\Im_m}\left\{\frac{1}{2\zeta_3}\Phi_{2j}^2\right\}\right\} > 0$$

and

$$\lim_{t\to+\infty}\Big[\|u(t)\|_2 + \|z(t)\|_2\Big] \le \sqrt{\frac{4\Phi}{\mathcal{H}-\mathcal{L}}},$$

where

$$\xi_{1i} = 2a_i + 2\xi - \varsigma_3 - \sum_{j=1}^{m}|v_{ij}|\mathcal{R}_j^k\varsigma_1 - \sum_{j=1}^{m}|w_{ij}|\mathcal{R}_j^k\varsigma_2 - \frac{1}{\zeta_1}\sum_{j=1}^{m}|p_{ji}|\mathcal{R}_i^l,\ \xi_{2i} = \sum_{j=1}^{m}\frac{|q_{ji}|\mathcal{R}_i^l}{\zeta_2},$$

$$\varepsilon_{1j} = 2b_j + 2\varepsilon - \zeta_3 - \sum_{i=1}^{n}|p_{ji}|\mathcal{R}_i^l\zeta_1 - \sum_{i=1}^{n}|q_{ji}|\mathcal{R}_i^l\zeta_2 - \frac{1}{\varsigma_1}\sum_{i=1}^{n}|v_{ij}|\mathcal{R}_j^k,\ \varepsilon_{2j} = \sum_{i=1}^{n}\frac{|w_{ij}|\mathcal{R}_j^k}{\varsigma_2},$$

$$\Phi_{1i} = \sum_{j=1}^{m}\Big(|v_{ij}| + |w_{ij}|\Big)\pi_j^k,\ \Phi_{2j} = \sum_{i=1}^{n}\Big(|p_{ji}| + |q_{ji}|\Big)\pi_i^l.$$

Proof. Consider the following Lyapunov function

$$G(t) = \frac{1}{2}u^T(t)u(t) + \frac{1}{2}z^T(t)z(t) \tag{12}$$

By virtue of Lemma 2.13 and assumption $\mathcal{A}_2$, we have

$$\begin{aligned}
D^\beta G(t) &\le \sum_{i=1}^{n}u_i(t)D^\beta u_i(t) + \sum_{j=1}^{m}z_j(t)D^\beta z_j(t)\\
&= \sum_{i=1}^{n}u_i(t)\Bigg\{-a_iu_i(t) + \sum_{j=1}^{m}v_{ij}\tilde{\alpha}_j(t) + \sum_{j=1}^{m}w_{ij}\tilde{\alpha}_j(t-\eta(t)) + L_i(t)\Bigg\}\\
&\quad + \sum_{j=1}^{m}z_j(t)\Bigg\{-b_jz_j(t) + \sum_{i=1}^{n}p_{ji}\tilde{\beta}_i(t) + \sum_{i=1}^{n}q_{ji}\tilde{\beta}_i(t-\eta(t)) + N_j(t)\Bigg\}\\
&\le -\sum_{i=1}^{n}a_iu_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{m}|v_{ij}||u_i(t)|\Big[\mathcal{R}_j^k|z_j(t)| + \pi_j^k\Big] + \sum_{i=1}^{n}\sum_{j=1}^{m}|w_{ij}||u_i(t)|\\
&\quad\times\Big[\mathcal{R}_j^k|z_j(t-\eta(t))| + \pi_j^k\Big] + \sum_{i=1}^{\hat{s}}M_i(t)|u_i(t)| - \sum_{j=1}^{m}b_jz_j^2(t) + \sum_{j=1}^{m}\sum_{i=1}^{n}|p_{ji}||z_j(t)|\\
&\quad\times\Big[\mathcal{R}_i^l|u_i(t)| + \pi_i^l\Big] + \sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}||z_j(t)|\Big[\mathcal{R}_i^l|u_i(t-\eta(t))| + \pi_i^l\Big] + \sum_{j=1}^{\check{s}}N_j|z_j(t)|\\
&\le -\sum_{i=1}^{n}a_iu_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{m}|v_{ij}|\mathcal{R}_j^k|u_i(t)||z_j(t)| + \sum_{i=1}^{n}\sum_{j=1}^{m}|v_{ij}|\pi_j^k|u_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{m}|w_{ij}|\mathcal{R}_j^k|u_i(t)|\\
&\quad\times|z_j(t-\eta(t))| + \sum_{i=1}^{n}\sum_{j=1}^{m}|w_{ij}|\pi_j^k|u_i(t)| + \sum_{i=1}^{\hat{s}}|u_i(t)|\Bigg[-\xi\, sgn\{u_i(t)\}\times\left[\frac{\sum_{i=1}^{n}|u_i(t)|}{\sum_{i=1}^{\hat{s}}|u_i(t)|}\right]\\
&\quad\times\sum_{j=1}^{n}|u_j(t)|\Bigg] - \sum_{j=1}^{m}b_jz_j^2(t) + \sum_{j=1}^{m}\sum_{i=1}^{n}|p_{ji}|\mathcal{R}_i^l|z_j(t)||u_i(t)| + \sum_{j=1}^{m}\sum_{i=1}^{n}|p_{ji}|\pi_i^l|z_j(t)|
\end{aligned}$$

$$+\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|\mathcal{R}_i^l|z_j(t-\eta(t))||u_i(t)|+\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|\pi_i^l|z_j(t)|$$

$$+\sum_{j=1}^{\check{s}}|z_j(t)|\Bigg[-\varepsilon sgn\{z_j(t)\}\Bigg[\frac{\sum_{j=1}^{m}|z_j(t)|}{\sum_{i=1}^{\check{s}}|z_j(t)|}\Bigg]\times\sum_{i=1}^{m}|z_i(t)|\Bigg]$$

$$\leq-\sum_{i=1}^{n}(a_i+\xi)u_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{m}|v_{ij}|\mathcal{R}_j^k|u_i(t)||z_j(t)|+\sum_{i=1}^{n}\sum_{j=1}^{m}|w_{ij}|\mathcal{R}_j^k|u_i(t)||z_j(t-\eta(t))|$$

$$+\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{m}\Big(|v_{ij}|+|w_{ij}|\Big)\pi_j^k\Bigg]|u_i(t)|-\sum_{j=1}^{m}(b_j+\varepsilon)z_j^2(t)+\sum_{j=1}^{m}\sum_{i=1}^{n}|p_{ji}|\mathcal{R}_i^l|z_j(t)||u_i(t)|$$

$$+\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|\mathcal{R}_i^l|z_j(t)||u_i(t-\eta(t))|+\sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n}\Big(|p_{ji}|+|q_{ji}|\Big)\pi_i^l\Bigg]|z_j(t)|$$

$$\leq-\sum_{i=1}^{n}(a_i+\xi)u_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{m}|v_{ij}|\mathcal{R}_j^k|u_i(t)||z_j(t)|+\sum_{i=1}^{n}\sum_{j=1}^{m}|w_{ij}|\mathcal{R}_j^k|u_i(t)||z_j(t-\eta(t))|$$

$$+\sum_{i=1}^{n}|u_i(t)|\Phi_{1i}-\sum_{j=1}^{m}(b_j+\varepsilon)z_j^2(t)+\sum_{j=1}^{m}\sum_{i=1}^{n}|p_{ji}|\mathcal{R}_i^l|z_j(t)||u_i(t)|$$

$$+\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|\mathcal{R}_i^l|z_j(t)||u_i(t-\eta(t))|+\sum_{j=1}^{m}|z_j(t)|\Phi_{2j}. \tag{13}$$

By means of important inequality, we get

$$|u_i(t)||z_j(t)|\leq\frac{\varsigma_1}{2}u_i^2(t)+\frac{1}{2\varsigma_1}z_j^2(t) \tag{14}$$

$$|u_i(t)||z_j(t-\eta(t))|\leq\frac{\varsigma_2}{2}u_i^2(t)+\frac{1}{2\varsigma_2}z_j^2(t-\eta(t)) \tag{15}$$

$$|u_i(t)|\Phi_{1i}\leq\frac{\varsigma_3}{2}u_i^2(t)+\frac{1}{2\varsigma_3}\Phi_{1i}^2 \tag{16}$$

$$|z_j(t)||u_i(t)|\leq\frac{\zeta_1}{2}z_j^2(t)+\frac{1}{2\zeta_1}u_i^2(t) \tag{17}$$

$$|z_j(t)||u_i(t-\eta(t))|\leq\frac{\zeta_2}{2}z_j^2(t)+\frac{1}{2\zeta_2}u_i^2(t-\eta(t)) \tag{18}$$

$$|z_j(t)|\Phi_{2j}\leq\frac{\zeta_3}{2}z_j^2(t)+\frac{1}{2\zeta_3}\Phi_{2j}^2. \tag{19}$$

Substituting (14)–(19) into (13), we get

$$D^\beta G(t)\leq-\sum_{i=1}^{n}(a_i+\xi)u_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{m}|v_{ij}|\mathcal{R}_j^k\Big[\frac{\varsigma_1}{2}u_i^2(t)+\frac{1}{2\varsigma_1}z_j^2(t)\Big]+\sum_{i=1}^{n}\sum_{j=1}^{m}|w_{ij}|\mathcal{R}_j^k\Big[\frac{\varsigma_1}{2}u_i^2(t)$$

$$+\frac{1}{2\varsigma_1}z_j^2(t-\eta(t))\Big]+\sum_{i=1}^{n}\Bigg[\frac{\varsigma_3}{2}u_i^2(t)+\frac{1}{2\varsigma_3}\Phi_{1i}^2\Bigg]$$

$$-\sum_{j=1}^{m}(b_j+\varepsilon)z_j^2(t)+\sum_{j=1}^{m}\sum_{i=1}^{n}|p_{ji}|\mathcal{R}_i^l\Big[\frac{\zeta_1}{2}z_j^2(t)+\frac{1}{2\zeta_1}u_i^2(t)\Big]+\sum_{j=1}^{m}\sum_{i=1}^{n}|q_{ji}|\mathcal{R}_i^l\Big[\frac{\zeta_1}{2}z_j^2(t)$$

$$+\frac{1}{2\zeta_1}u_i^2(t-\eta(t))\Big]+\sum_{j=1}^{m}\Bigg[\frac{\zeta_3}{2}z_j^2(t)+\frac{1}{2\zeta_3}\Phi_{2j}^2\Bigg]$$

$$\leq\sum_{i=1}^{n}\Bigg\{-a_i-\xi+\frac{\varsigma_3}{2}+\frac{1}{2}\sum_{j=1}^{m}|v_{ij}|\mathcal{R}_j^k\varsigma_1+\frac{1}{2}\sum_{j=1}^{m}|w_{ij}|\mathcal{R}_j^k\varsigma_2+\frac{1}{2\zeta_1}\sum_{j=1}^{m}|p_{ji}|\mathcal{R}_i^l\Bigg\}u_i^2(t)$$

$$+\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}\frac{1}{2\varsigma_2}|q_{ji}|\mathcal{R}_i^l\right\}u_i^2(t-\eta(t))+\frac{1}{2\varsigma_3}\sum_{i=1}^{n}\Phi_{1i}^2$$

$$+\sum_{j=1}^{m}\left\{-b_j-\varepsilon+\frac{\zeta_3}{2}+\frac{1}{2}\sum_{i=1}^{n}|p_{ji}|\mathcal{R}_i^l\zeta_1+\frac{1}{2}\sum_{i=1}^{n}|q_{ji}|\mathcal{R}_i^l\zeta_2+\frac{1}{2\varsigma_1}\sum_{i=1}^{n}|v_{ij}|\mathcal{R}_j^k\right\}z_j^2(t)$$

$$+\sum_{j=1}^{m}\left\{\sum_{i=1}^{n}\frac{1}{2\varsigma_2}|w_{ij}|\mathcal{R}_j^k\right\}z_j^2(t-\eta(t))+\frac{1}{2\zeta_3}\sum_{j=1}^{m}\Phi_{2j}^2$$

$$\leq -\min_{i\in\Im_n}\{\xi_{1i}\}\sum_{i=1}^{n}\frac{u_i^2(t)}{2}+\min_{i\in\Im_n}\{\xi_{2i}\}\sum_{i=1}^{n}\frac{u_i^2(t-\eta(t))}{2}$$

$$-\min_{i\in\Im_n}\{\varepsilon_{1j}\}\sum_{j=1}^{m}\frac{z_j^2(t)}{2}+\min_{i\in\Im_n}\{\varepsilon_{2j}\}\sum_{j=1}^{m}\frac{z_j^2(t-\eta(t))}{2}+\Phi$$

$$\leq -\mathcal{H}G(t)+\mathcal{L}G(t-\eta(t))+\Phi. \tag{20}$$

Let $\Lambda(t)=G(t)-\frac{\Phi}{\mathcal{H}-\mathcal{L}}$, then we have

$$\begin{aligned}D^{\beta}\Lambda(t)&\leq -\mathcal{H}\Lambda(t)+\mathcal{L}\Lambda(t-\eta(t))\\&\leq -\mathcal{H}\Lambda(t)+\mathcal{L}\max_{t-\eta\leq\omega\leq t}\Lambda(\omega),\end{aligned} \tag{21}$$

where $\eta(t)\in[0,\eta]$. According to Lemma 2.15, there exist a constants ψ_1 and ψ_2 such that

$$\Lambda(t)\leq\psi_1\mathcal{E}_{\beta}\left(-\psi_2 t^{\beta}\right),\ t\geq t^*,$$

which implies that

$$G(t)\leq\psi_1\mathcal{E}_{\beta}\left(-\psi_2 t^{\beta}\right)+\frac{\Phi}{\mathcal{H}-\mathcal{L}}. \tag{22}$$

Note that

$$G(t)=\frac{1}{2}u^T(t)u(t)+\frac{1}{2}z^T(t)z(t)=\frac{1}{2}\left[\|u(t)\|^2+\|z(t)\|^2\right].$$

According to Lemma 2.14, we have

$$\frac{1}{2}\left[\|u(t)\|+\|z(t)\|\right]^2\leq\|u(t)\|^2+\|z(t)\|^2\leq 2\psi_1\mathcal{E}_{\beta}\left(-\psi_2 t^{\beta}\right)+\frac{2\Phi}{\mathcal{H}-\mathcal{L}}$$

which shows that

$$\|u(t)\|+\|z(t)\|\leq\sqrt{4\psi_1\mathcal{E}_{\beta}\left(-\psi_2 t^{\beta}\right)+\frac{4\Phi}{\mathcal{H}-\mathcal{L}}}.$$

Based on above inequality and Remark 2.16, we obtain

$$\lim_{t\to\infty}\left[\|u(t)\|_2+\|z(t)\|_2\right]\leq\sqrt{\frac{4\Phi}{\mathcal{H}-\mathcal{L}}}.$$

Therefore, drive-system (1) and controlled response system (6) realize quasi-synchronized via pinning controller (11). The proof is completed. □

Remark 3.2. Although, the quasi-synchronization of fractional order neural networks (FNNs) with state feedback control have been extensively studied in the existing work of literatures, see for Refs. [44,45]. As far as we know, there are no results about the combination of quasi-synchronization and pinning control analysis in all kinds of neural network models, especially BAM neural networks of such model. In view of this, pinning control for fractional order synchronization of BAM neural network model is presented and its quasi-synchronization is studied.

Remark 3.3. In practice, a synchronization error signal can be controlled in a certain range and synchronization error bound closely related to control gains and number of randomly selecting pinned neurons. To illuminate how to design a suitable pinning controller in application perspective to achieve quasi-synchronization or stabilization, we take Theorem 3.2 for instance, we are able to design following steps:

4. Stabilization results

In this part, we will derive the existence of equilibrium point for FBAMNNs with discontinuous activations and time-varying delays based on the Kakutani's fixed point theorem for set-valued map analysis (Table 1).

Table 1
The Algorithm to design the novel pinning controller.

Algorithm	
step.1:	Initialize the system parameters A, B, P, Q, U, V, I, J.
step.2:	Randomly selecting $\hat{s}$ of pinned neurons from all neurons in one layer, similarly $\check{s}$ of pinned neurons in another layer.
step.3:	Properly find out the values of $\mathcal{R}_j^k$, $\mathcal{R}_i^l$, π_j^k, π_i^l according to assumptions $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$.
step.4:	Choose the control strengths ξ and ε.
step.5:	Given $\varsigma_1, \varsigma_2, \varsigma_3, \zeta_1, \zeta_2, \zeta_3$ and compute to easily obtain $\mathcal{H}$, $\mathcal{L}$, Φ.
step.6:	Check whether $\mathcal{H} > \mathcal{L} > 0$ and $\Phi > 0$. If success, the procedure further moves to next level. Otherwise the procedure turns back to adjust the control strengths in step 4.
step.7:	Based on the proper control strengths, we design a novel pinning control.

4.1. Existence of equilibrium point via Kakutani's fixed point theorem

Theorem 4.1. *Under the assumptions $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$, then the FBAMNNs with discontinuous activations (1) has at least one equilibrium point.*

Proof. Let X be a Banach space and a norm is defined by $\|\kappa\|_1 = \sum_{f=1}^{n+m} |k_f|$, $\forall\ (\kappa_1, \ldots, \kappa_n, \kappa_{n+1}, \ldots, \kappa_{n+m})^T \in X$. Obviously, existence of equilibrium point for the system (1) is equivalent to the following differential inclusion system.

$$\begin{cases} k_i(t) \in \frac{1}{a_i}\Big[\sum_{j=1}^m v_{ij}\overline{co}\{h_j(l_j(t))\} + \sum_{j=1}^m w_{ij}\overline{co}\{h_j(l_j(t-\eta(t)))\} + J_i\Big] \\ l_j(t) \in \frac{1}{b_j}\Big[\sum_{i=1}^n p_{ji}\overline{co}\{g_i(k_i(t))\} + \sum_{i=1}^n q_{ji}\overline{co}\{g_i(k_i(t-\eta(t)))\} + I_j\Big] \end{cases} \tag{23}$$

for $i \in \Im_n$ and $j \in \Im_m$. Then by using Kakutani's fixed point theorem, in order to we will prove the existence of equilibrium point in three steps.

Step: 1. From assumption $[\mathcal{A}_1]$, the neuron activations f_j and g_i are bounded. Denote a compact convex subset is

$$E = \Big\{\kappa = \big(k_1, \ldots, k_n, l_1, \ldots, l_m\big)^T \in X : \|\kappa\|_1 \le \varpi\Big\},$$

where

$$\varpi = \sum_{i=1}^n \sum_{j=1}^m \frac{\big(|v_{ij}| + |w_{ij}|\big)\varpi_j^k}{a_i} + \sum_{j=1}^m \sum_{i=1}^n \frac{\big(|p_{ji}| + |q_{ji}|\big)\varpi_i^l}{b_j} + \sum_{i=1}^n \frac{J_i}{a_i} + \sum_{j=1}^m \frac{I_j}{b_j}. \tag{24}$$

Step: 2. Now, we define a set-valued map $\Omega(\kappa) = \Big(\Omega_1(k_1), \ldots, \Omega_n(k_n), \Omega_1(l_1), \ldots, \Omega_m(l_m)\Big)^T : X \to P_{Kc}(X)$. Based on the aforementioned argument for system (1), $\Omega(\kappa)$ is an upper semi-continuous with non-empty compact convex values.

Step: 3. By means of differential inclusion theory and set-valued map analysis, there exist $\lambda_j(t) \in \overline{co}\{h_j(l_j(t))\}$ and $\mu_i(t) \in \overline{co}\{g_i(k_i(t))\}$, such that $\varrho = \big(\breve{\varrho}_1, \ldots, \breve{\varrho}_n, \tilde{\varrho}_1, \ldots, \tilde{\varrho}_m\big)^T \in \Omega(\kappa)$, where

$$\begin{cases} \breve{\varrho}_i = \frac{1}{a_i}\Big[\sum_{j=1}^m v_{ij}\lambda_j(t) + \sum_{j=1}^m w_{ij}\lambda_j(t-\eta(t)) + J_i\Big] \\ \tilde{\varrho}_j = \frac{1}{b_j}\Big[\sum_{i=1}^n p_{ji}\mu_i(t) + \sum_{i=1}^n q_{ji}\mu_i(t-\eta(t)) + I_j\Big]. \end{cases}$$

Then

$$\begin{aligned} \|\varrho\|_1 &= \sum_{f=1}^{n+m} |\varrho_f| = \sum_{i=1}^n |\breve{\varrho}_i| + \sum_{j=1}^m |\tilde{\varrho}_j| \\ &= \sum_{i=1}^n \left|\left\{\sum_{j=1}^m \frac{1}{a_i} v_{ij}\lambda_j(t) + \sum_{j=1}^m \frac{1}{a_i} w_{ij}\lambda_j(t-\eta(t)) + \frac{J_i}{a_i}\right\}\right| \\ &\quad + \sum_{i=1}^n \left|\left\{\sum_{i=1}^n \frac{1}{b_j} p_{ji}\mu_i(t) + \sum_{i=1}^n \frac{1}{b_j} q_{ji}\mu_i(t-\eta(t)) + \frac{I_j}{b_j}\right\}\right| \\ &\le \sum_{i=1}^n \left\{\sum_{j=1}^m \frac{1}{a_i}|v_{ij}|\varpi_j^k + \sum_{j=1}^m \frac{1}{a_i}|w_{ij}|\varpi_j^k + |\frac{J_i}{a_i}|\right\} \\ &\quad + \sum_{i=1}^n \left\{\sum_{i=1}^n \frac{1}{b_j}|p_{ji}|\varpi_i^l + \sum_{i=1}^n \frac{1}{b_j} q_{ji}\varpi_i^l + |\frac{I_j}{b_j}|\right\} \\ &= \varpi, \end{aligned}$$

where Assumption $[\mathcal{A}_1]$ has been used. Thus for any $\kappa \in E$ and $\varrho \in \Omega(\kappa)$, we have $\varrho \in E$. Based on Lemma 2.11 that, a map Ω: $E \to P_{Kc}(E)$ has at least one fixed point $\kappa^* = \big(k_1^*, \ldots, k_n^*, l_1^*, \ldots, l_m^*\big)^T \in \Omega(k^*)$, it follows there exist al least one equilibrium point for FBAMNNs with discontinuous activations (1). Hence the proof is completed. □

4.2. β-exponential stabilization under nonlinear pinning control

In this segment, we need to design the most effective controller to stabilize unstable equilibrium point of FBANNS (1) to origin. By using change of variables $x(t)=k(t)-k^*$ and $y(t)=l(t)-l^*$, new fractional order error system can be derived from (1):

$$\begin{cases} D^\beta x_i(t) = -a_i x_i(t) + \sum_{j=1}^m v_{ij}\Big[h_j(l_j(t)) - h_j(l_j^*)\Big] + \sum_{j=1}^m w_{ij}\Big[h_j(l_j(t-\eta(t))) - h_j(l_j^*)\Big] \\ D^\beta y_j(t) = -b_j y_j(t) + \sum_{i=1}^n p_{ji}\Big[g_i(k_i(t)) - g_i(k_i^*)\Big] + \sum_{i=1}^n q_{ji}\Big[g_i(k_i(t-\eta(t))) - g_i(k_i^*)\Big]. \end{cases} \tag{25}$$

Due to discontinuity of vector fields, classical pinning control strategy is very difficult to achieve stabilization goal. In this particular case, a discontinuous pinning control strategy is needed to stabilize unstable equilibrium point of FBANNS (1), and the controlled system can be expressed by the following fractional order discontinuous system.

$$\begin{cases} D^\beta x_i(t) = -a_i x_i(t) + \sum_{j=1}^m v_{ij}\Big[h_j(l_j(t)) - h_j(l_j^*)\Big] + \sum_{j=1}^m w_{ij}\Big[h_j(l_j(t-\eta(t))) - h_j(l_j^*)\Big] + E_i(t) \\ D^\beta y_j(t) = -b_j y_j(t) + \sum_{i=1}^n p_{ji}\Big[g_i(k_i(t)) - g_i(k_i^*)\Big] + \sum_{i=1}^n q_{ji}\Big[g_i(k_i(t-\eta(t))) - g_i(k_i^*)\Big] + H_j(t) \end{cases} \tag{26}$$

for $i \in \Im_n$ and $j \in \Im_m$, where $E_i(t)$ and $H_j(t)$ are pinning controllers which is designed by

$$\begin{cases} E_i(t) = E_{1i}(t) - \tau sgn\{k_i(t)-k^*\}, \ i \in \Im_n \\ H_j(t) = H_{1j}(t) - \theta sgn\{l_j(t)-l^*\}, \ j \in \Im_m, \end{cases} \tag{27}$$

where

$$E_{1i}(t) = \begin{cases} -\xi sgn\{k_i(t)-k^*\} \times \left[\frac{\sum_{i=1}^n |k_i(t)-k^*|}{\sum_{i=1}^{\hat{s}} |k_i(t)-k^*|}\right] \times \sum_{j=1}^n |k_j(t)-k^*| & \text{if } i \in \Im_{\hat{s}} = \{1,2,\ldots,\hat{s}\} \\ 0 & \text{if } i \in \Im_n \backslash \Im_{\hat{s}} = \{\hat{s}+1,\ldots,n\} \end{cases}$$

$$H_{1j}(t) = \begin{cases} -\varepsilon sgn\{l_j(t)-l^*\} \times \left[\frac{\sum_{j=1}^m |l_j(t)-l^*|}{\sum_{i=1}^{\check{s}} |l_j(t)-l^*|}\right] \times \sum_{i=1}^m |l_i(t)-l^*| & \text{if } j \in \Im_{\check{s}} = \{1,2,\ldots,\check{s}\} \\ 0 & \text{if } j \in \Im_m \backslash \Im_{\check{s}} = \{\check{s}+1,\ldots,m\}, \end{cases}$$

ξ, ε, τ and θ are adjustable positive constants.

By means of set-valued map analysis and differential inclusion theory, from (26)

$$\begin{cases} D^\beta x_i(t) \in & -a_i x_i(t) + \sum_{j=1}^m v_{ij}\Big[\overline{co}\{h_j(l_j(t))\} - \overline{co}\{h_j(l_j^*)\}\Big] + \sum_{j=1}^m w_{ij}\Big[\overline{co}\{h_j(l_j(t-\eta(t)))\} \\ & -\overline{co}\{h_j(l_j^*)\}\Big] + E_i(t) \\ D^\beta y_j(t) \in & -b_j y_j(t) + \sum_{i=1}^n p_{ji}\Big[\overline{co}\{g_i(k_i(t))\} - \overline{co}\{g_i(k_i^*)\}\Big] + \sum_{i=1}^n q_{ji}\Big[\overline{co}\{g_i(k_i(t-\eta(t)))\} \\ & -\overline{co}\{g_i(k_i^*)\}\Big] + H_j(t) \end{cases} \tag{28}$$

for a.a $t \geq 0$. Or there exist $\lambda_j(t) \in \overline{co}\{h_j(l_j(t))\}$, $\lambda_j^* \in \overline{co}\{h_j(l_j^*)\}$, $\mu_i(t) \in \overline{co}\{g_i(k_i(t))\}$ and $\mu_i^* \in \overline{co}\{g_i(k_i^*)\}$ such that

$$\begin{cases} D^\beta x_i(t) = -a_i x_i(t) + \sum_{j=1}^m v_{ij}\tilde{\lambda}_j(t) + \sum_{j=1}^m w_{ij}\tilde{\lambda}_j(t-\eta(t)) + E_i(t) \\ D^\beta y_j(t) = -b_j y_j(t) + \sum_{i=1}^n p_{ji}\tilde{\mu}_i(t) + \sum_{i=1}^n q_{ji}\tilde{\mu}_i(t-\eta(t)) + H_j(t), \end{cases} \tag{29}$$

a.a. $t \in [0, T)$, where

$$\tilde{\lambda}_j(t) = \lambda_j(t) - \lambda_j^* \in \overline{co}\{h_j(l_j(t)\} - \overline{co}\{h_j(l_j^*)\} \tag{30}$$

and

$$\tilde{\mu}_i(t) = \mu_i(t) - \mu_i^* \in \overline{co}\{g_i(k_i(t)\} - \overline{co}\{g_i(k_i^*)\} \tag{31}$$

.

Next, β-exponential stabilization results of time-varying delayed FBAMNNs with discontinuous activations can be provided.

Theorem 4.2. *Assuming the conditions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold, and then equilibrium point for FBAMNNs with discontinuous activations (1) is β-exponentially stable if there exist a diagonal matrices $\Lambda = diag\Big(\lambda_1,\ldots,\lambda_n\Big) > 0$, $\Upsilon = diag\Big(\gamma_1,\ldots,\gamma_m\Big) > 0$, Φ_1, Φ_2, Ψ_1, Ψ_2 and four positive constants $\varsigma, \zeta, \delta, \mu$, such that the following LMI:*

(i)

$$\begin{bmatrix} -2\Lambda A - 2\Lambda\xi + \Psi_1\hat{\mathcal{R}}^2 + \varsigma\Lambda & \Lambda|V| & \Lambda|W| \\ \star & -\Phi_1 & 0 \\ \star & \star & -\Phi_2 \end{bmatrix} < 0,$$

$$\begin{bmatrix} -2\Upsilon B - 2\Upsilon\varepsilon + \Phi_1\check{\mathcal{R}}^2 + \zeta\Upsilon & \Upsilon|P| & \Upsilon|Q| \\ \star & -\Psi_1 & 0 \\ \star & \star & -\Psi_2 \end{bmatrix} < 0,$$

(ii) $\Psi_2\hat{\mathcal{R}}^2 - \delta\Lambda < 0,\ \Phi_2\check{\mathcal{R}}^2 - \mu\Upsilon < 0,$

(iii) $\min\{\varsigma, \zeta\} > \max\{\delta, \mu\} > 0,$

where $\hat{R} = diag\left(\mathcal{R}_1^l, \ldots, \mathcal{R}_n^l\right)$ *and* $\check{R} = diag\left(\mathcal{R}_1^k, \ldots, \mathcal{R}_m^k\right)$. *Furthermore, the control gains are subjected to*

$$\tau > \max_{i\in\Im_n}\left\{\sum_{j=1}^{m}\left(|v_{ij}| + |w_{ij}|\right)\pi_j^k\right\} \text{ and } \theta > \max_{j\in\Im_m}\left\{\sum_{i=1}^{n}\left(|p_{ji}| + |q_{ji}|\right)\pi_i^l\right\}.$$

Proof. Consider the following Lyapunov-functional

$$G(t) = |x(t)|^T\Lambda|x(t)| + |y(t)|^T\Upsilon|y(t)| \tag{30}$$

where $|x(t)| = \left(|x_1(t)|, \ldots, |x_n(t)|\right)^T$ and $|y(t)| = \left(|y_1(t)|, \ldots, |y_m(t)|\right)^T$.

From Assumption $[\mathcal{A}_1] - [\mathcal{A}_2]$, Lemmas 2.13 and 2.17, we have

$$\begin{aligned} D^\beta G(t) &= 2|x(t)|^T\Lambda D^\beta|x(t)| + 2|y(t)|^T\Upsilon D^\beta|y(t)| \\ &= 2\sum_{i=1}^{n}|x(t)|\lambda_i D^\beta|x(t)| + 2\sum_{j=1}^{m}|y(t)|\gamma_j D^\beta|y(t)| \\ &\le -2\sum_{i=1}^{n}|x_i(t)|\lambda_i a_i|x_i(t)| + 2\sum_{i=1}^{n}\sum_{j=1}^{m}|x_i(t)|\lambda_i|v_{ij}|\,|\tilde{\lambda}_j(t)| + 2\sum_{i=1}^{n}\sum_{j=1}^{m}|x_i(t)|\lambda_i|w_{ij}|\,|\tilde{\lambda}_j(t-\eta(t))| \\ &\quad +2\sum_{i=1}^{n}|x_i(t)|\lambda_i\left[-\xi sgn\{x_i(t)\}\times\left[\frac{\sum_{i=1}^{n}|x_i(t)}{\sum_{i=1}^{\hat{s}}|x_i(t)|}\right]\times\sum_{j=1}^{n}|x_j(t)|\right] - 2\sum_{i=1}^{n}|x_i(t)|\lambda_i\tau\hat{\epsilon}_i(t) \\ &\quad -2\sum_{j=1}^{m}|y_j(t)|\gamma_j b_j|y_j(t)| + 2\sum_{j=1}^{m}\sum_{i=1}^{n}|y_j(t)|\gamma_j|v_{ij}|\,|\tilde{\mu}_i(t)| + 2\sum_{j=1}^{m}\sum_{i=1}^{n}|y_j(t)|\gamma_j|w_{ij}|\,|\tilde{\mu}_i(t-\eta(t))| \\ &\quad +2\sum_{j=1}^{m}|y_j(t)|\gamma_j\left[-\varepsilon sgn\{y_j(t)\}\times\left[\frac{\sum_{j=1}^{m}|y_j(t)|}{\sum_{i=1}^{\check{s}}|y_j(t)|}\right]\times\sum_{i=1}^{m}|y_i(t)|\right] - 2\sum_{j=1}^{m}|y_j(t)|\gamma_j\theta\check{\epsilon}_j(t) \\ &\le -2\sum_{i=1}^{n}|x_i(t)|\lambda_i\left(a_i+\xi\right)|x_i(t)| + 2\sum_{i=1}^{n}\sum_{j=1}^{m}|x_i(t)|\lambda_i|v_{ij}|\left[\mathcal{R}_j^k\left|y_j(t)\right| + \pi_j^k\right] \\ &\quad +2\sum_{i=1}^{n}\sum_{j=1}^{m}|x_i(t)|\lambda_i|w_{ij}|\left[\mathcal{R}_j^k\left|y_j(t-\eta(t))\right| + \pi_j^k\right] - 2\sum_{i=1}^{n}\lambda_i\tau|x_i(t)| \\ &\quad -2\sum_{j=1}^{m}|y_j(t)|\gamma_j\left(b_j+\varepsilon\right)|y_j(t)| + 2\sum_{j=1}^{m}\sum_{i=1}^{n}|y_j(t)|\gamma_j|v_{ij}|\left[\mathcal{R}_i^l\left|u_i(t)\right| + \pi_i^l\right] \\ &\quad +2\sum_{j=m}^{n}\sum_{i=1}^{n}|y_j(t)|\gamma_j|w_{ij}|\left[\mathcal{R}_i^l\left|x_i(t-\eta(t))\right| + \pi_i^l\right] - 2\sum_{j=1}^{m}|y_j(t)|\gamma_j\theta, \end{aligned} \tag{31}$$

where $\hat{\epsilon}_i(t) \in sgn\{x_i(t)\}$ and $\hat{\epsilon}_j(t) \in sgn\{y_j(t)\}$. According to theorem conditions with respect to control gains, we have

$$-2\sum_{i=1}^{n}\lambda_i\tau|x_i(t)| + 2\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\left(|v_{ij}| + |w_{ij}|\right)\pi_j^k|x_i(t)| \le 0$$

$$-2\sum_{j=1}^{m}\gamma_j\theta|y_j(t)| + 2\sum_{j=1}^{m}\sum_{i=1}^{n}\gamma_j\left(|p_{ij}| + |q_{ij}|\right)\pi_i^l|y_j(t)| \le 0.$$

From (31) is equivalent to the following vector form:

$$\begin{aligned} D^\beta G(t) &\le -2|x(t)|^T\Lambda\left(A+\xi\right)|x(t)| + 2|x(t)|^T\Lambda|V|\check{\mathcal{R}}|y(t)| + 2|x(t)|^T\Lambda|W|\check{\mathcal{R}}|y(t-\eta(t))| \\ &\quad -2|y(t)|^T\Upsilon\left(B+\varepsilon\right)|y(t)| + 2|y(t)|^T\Upsilon|P|\hat{\mathcal{R}}|x(t)| + 2|y(t)|^T\Upsilon|P|\hat{\mathcal{R}}|x(t-\eta(t))| \end{aligned}$$

$$\le |x(t)|^T\Big[-2\Lambda A-2\Lambda\xi+\Lambda|V|\Phi_1^{-1}|V|^T\Lambda+\Lambda|W|\Phi_2^{-1}|W|^T\Lambda+\Psi_1\hat{\mathcal{R}}^2+\varsigma\Big]|x(t)|$$
$$-\varsigma|x(t)|^T\Lambda|x(t)|+|x(t-\eta(t))|^T\Big[\Psi_2\hat{\mathcal{R}}^2-\delta\Lambda\Big]|x(t-\eta(t))|+\delta|x(t-\eta(t))|^T\Lambda|x(t-\eta(t))|$$
$$+|y(t)|^T\Big[-2\Upsilon B-2\Upsilon\varepsilon+\Upsilon|P|\Psi_1^{-1}|P|^T\Upsilon+\Upsilon|Q|\Psi_2^{-1}|Q|^T\Upsilon++\Phi_1\check{\mathcal{R}}^2+\zeta\Upsilon\Big]|y(t)|$$
$$-\zeta|y(t)|^T\Upsilon|y(t)|+|y(t-\eta(t))|^T\Big[\Phi_2\check{\mathcal{R}}^2-\mu\Upsilon\Big]|y(t-\eta(t))|+\mu|y(t-\eta(t))|^T\Upsilon|y(t-\eta(t))|, \tag{32}$$

where Lemma 2.17 has been used.

By virtue of condition (i) and (ii) of Theorem 4.2 and (32), one has

$$D^\beta G(t)\le-\varsigma|x(t)|^T\Lambda|x(t)|+\delta|x(t-\eta(t))|^T\Lambda|x(t-\eta(t))|$$
$$-\zeta|y(t)|^T\Upsilon|y(t)|+\mu|y(t-\eta(t))|^T\Upsilon|y(t-\eta(t))|$$
$$\le-\alpha_1G(t)+\alpha_2\sup_{\omega\in[t-\eta,0]}G(\omega), \tag{33}$$

where $\alpha_1=\min\{\varsigma,\zeta\}$, $\alpha_2=\max\{\delta,\mu\}$ and $\eta(t)\in[0,\eta.]$ Now for any solution $(x^T(t),y^T(t))^T$ of (25) which satisfy the Razumikhin condition [27].

$$G(\omega)\le G(t),\ \omega\in[t-\eta,0]. \tag{34}$$

According to condition (iii) of Theorem 4.2, there exist a positive constant $\varrho>0$ such that $0<\varrho\le\alpha_1-\alpha_2$. Combining (33) and (34), we have

$$D^\beta G(t)\le-\varrho G(t). \tag{35}$$

From the famous Gronwall-inequality, we have

$$G(t)\le G(0)\exp\Big(\int_0^t\frac{-\varrho}{\Gamma(\beta)}(t-\omega)^{\beta-1}d\omega\Big)$$
$$=G(0)\exp\Big(\frac{\varrho}{\Gamma(\beta+1)}t^\beta\Big). \tag{36}$$

Otherwise, the Lyapunov functional G(t) satisfies

$$\lambda^*_{\min}(\Lambda)\|x(t)\|^2+\lambda^*_{\min}(\Upsilon)\|y(t)\|^2\le G(t)\le\lambda^*_{\max}(\Lambda)\|x(t)\|^2+\lambda^*_{\max}(\Upsilon)\|y(t)\|^2,$$

which implies that

$$\mathcal{K}_{\min}\Big[\|x(t)\|^2+\|y(t)\|^2\Big]\le G(t)\le\mathcal{K}_{\min}\Big[\|x(t)\|^2+\|y(t)\|^2\Big], \tag{37}$$

where $\mathcal{K}_{\min}=\min\Big\{\lambda^*_{\min}(\Lambda),\lambda^*_{\min}(\Upsilon)\Big\}$ and $\mathcal{K}_{\max}=\max\Big\{\lambda^*_{\max}(\Lambda),\lambda^*_{\max}(\Upsilon)\Big\}$.

On the otherhand,

$$G(0)\le\lambda^*_{\max}(\Lambda)\|x(0)\|^2+\lambda^*_{\max}(\Upsilon)\|y(0)\|^2$$
$$=\mathcal{K}_{\min}\Big[\|x(0)\|^2+\|y(0)\|^2\Big]. \tag{38}$$

Combining Eq. (36), (37) and (38), we obtain

$$\|x(t)\|^2+\|y(t)\|^2\le\frac{\mathcal{K}_{\max}}{\mathcal{K}_{\min}}\Big[\|x(0)\|^2+\|y(0)\|^2\Big]\exp\Big(\frac{-\varrho}{\Gamma(\beta+1)}t^\beta\Big). \tag{39}$$

By using Lemma 2.14, we obtain

$$\|k(t)-k^*\|+\|l(t)-l^*\|\le\tilde{\mathcal{K}}\Big[\|k(0)-k^*\|+\|l(0)-l^*\|\Big]\exp\Big(\frac{\varrho}{2\Gamma(\beta+1)}t^\beta\Big)$$
$$\le\tilde{\mathcal{K}}\sup_{\omega\in[-\eta,0]}\Big[\|k(\omega)-k^*\|+\|l(\omega)-l^*\|\Big]\exp\Big(\frac{-\varrho}{2\Gamma(\beta+1)}t^\beta\Big), \tag{40}$$

where $\tilde{\mathcal{K}}=\frac{\mathcal{K}_{\max}}{\mathcal{K}_{\min}}$. From Definition 2.7, equilibrium point for FBAMNNs (1) is β-exponentially stable. This ends the proof. □

Remark 4.3. Chen et al. [34] dealt with the synchronization condition of fractional order delayed neural networks with discontinuous neuron activation by using adaptive feedback controller including discontinuous terms. Zhixia et al. [37] investigated the global stabilization in finite-time fractional order neural networks with discontinuous neuron activation by simple feedback controller including discontinuous terms. It is seen that all the aforementioned controllers are applied to every neuron of FNNs, which could be very high priced and

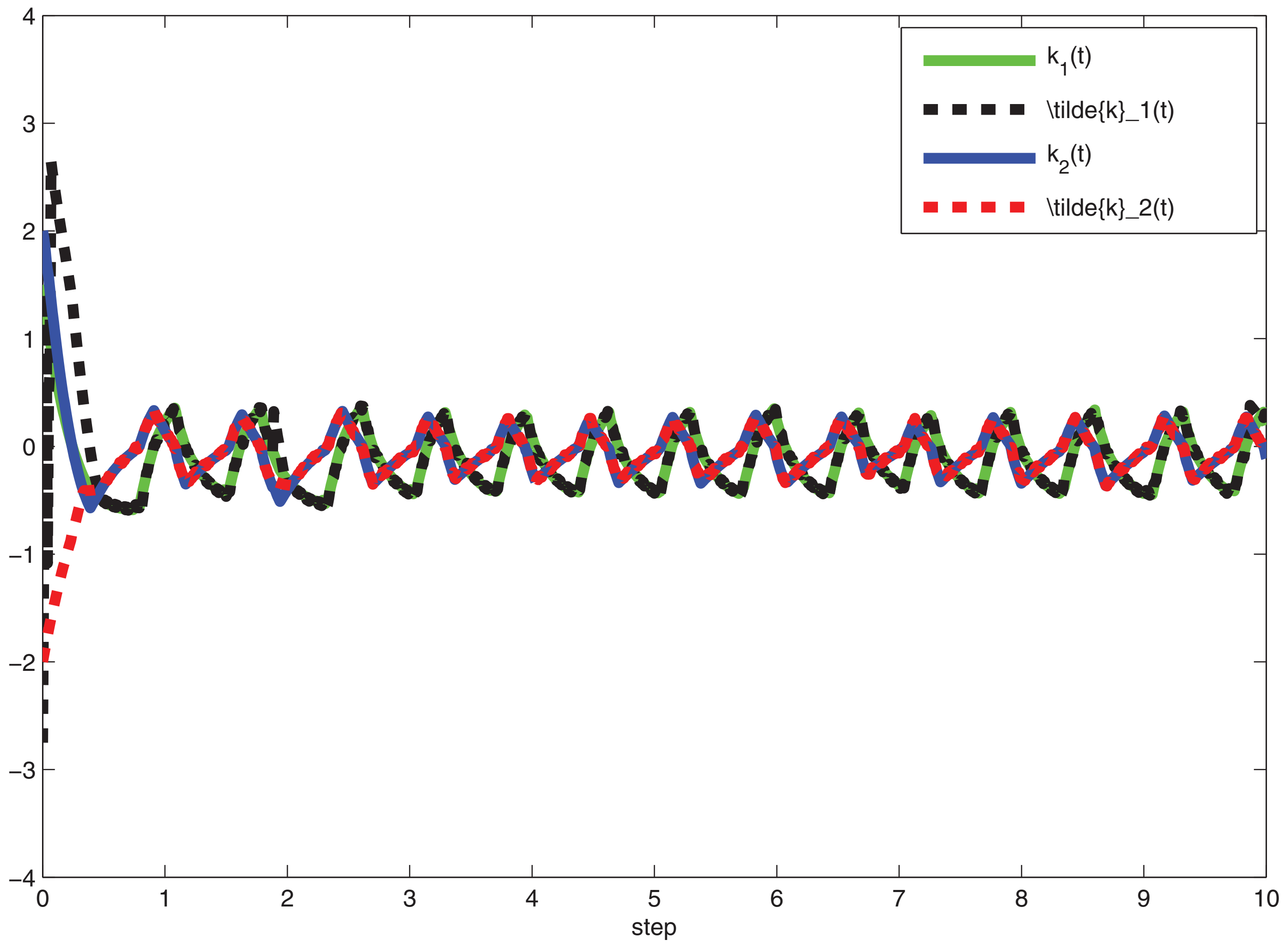

Fig. 1. The state trajectories of $k_i(t)$ vs. $\tilde{k}_i(t)$ for $i = 1, 2$.

impractical. Different from the control methods used in [34,37], designed pinning control methods are more effective because it has been applied to one neuron or the huge number of neurons instead of all neurons, which greatly reduce the control costs and consumption.

Remark 4.4. Reviewing the previous works in the literature, very few results of stabilization for fractional order delayed neural networks with discontinuous(or continuous) activations have been concerned [19,29,37,56]. However, there are no results at present to concern the stabilization problem for fractional-delayed order BAM neural networks with discontinuous activations via pinning control as far as we know. So the presented β-exponential result here is new.

5. Existence of solution in the Filippov sense

In order to further proceed and sake of simplicity, we first introduce some notations. Let

$$y(t) = \begin{bmatrix} k(t) \\ l(t) \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} 0 & V \\ P & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} 0 & W \\ Q & 0 \end{bmatrix}, \quad f(y(t)) = \begin{bmatrix} g(k(t)) \\ h(l(t)) \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} J \\ I \end{bmatrix}.$$

FBAMNNs system (1) is equivalent to the following form:

$$\begin{cases} D^{\beta} y(t) = -\mathcal{B}y(t) + \mathcal{K}f(y(t)) + \mathcal{S}f(y(t-\eta(t))) + \mathcal{N} \\ y(\omega) = \Psi(\omega) = \left(\phi^T(\omega), \psi^T(\omega)\right)^T, \ \omega \in [-\eta, 0], \end{cases} \tag{41}$$

and the norm of initial conditions is defined as $\|\Psi\| = \sup_{\omega\in[-\eta,0]} \|\Psi(\omega)\|$.

Theorem 5.1. *Under the assumptions $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$, then there exists at least one solution $(k^T(t), l^T(t))^T$ of FBAMNNs with discontinuous activations (1) on $[0, +\infty)$ in the sense of Filippov.*

Proof. By means of brief account in Section 2, the set valued map $y(t) \hookrightarrow -\mathcal{B}y(t) + \mathcal{K}\mathcal{F}(y(t)) + \mathcal{S}\mathcal{F}(y(t-\eta(t))) + \mathcal{N}$ is upper semi continuous with non empty, bounded, compact and closed convex values. Hence the local existence of solution $y(t)$ for (41) with initial value $\Phi(\omega)$ can be guaranteed. By virtue of property (1) in Definition 2.4, one has

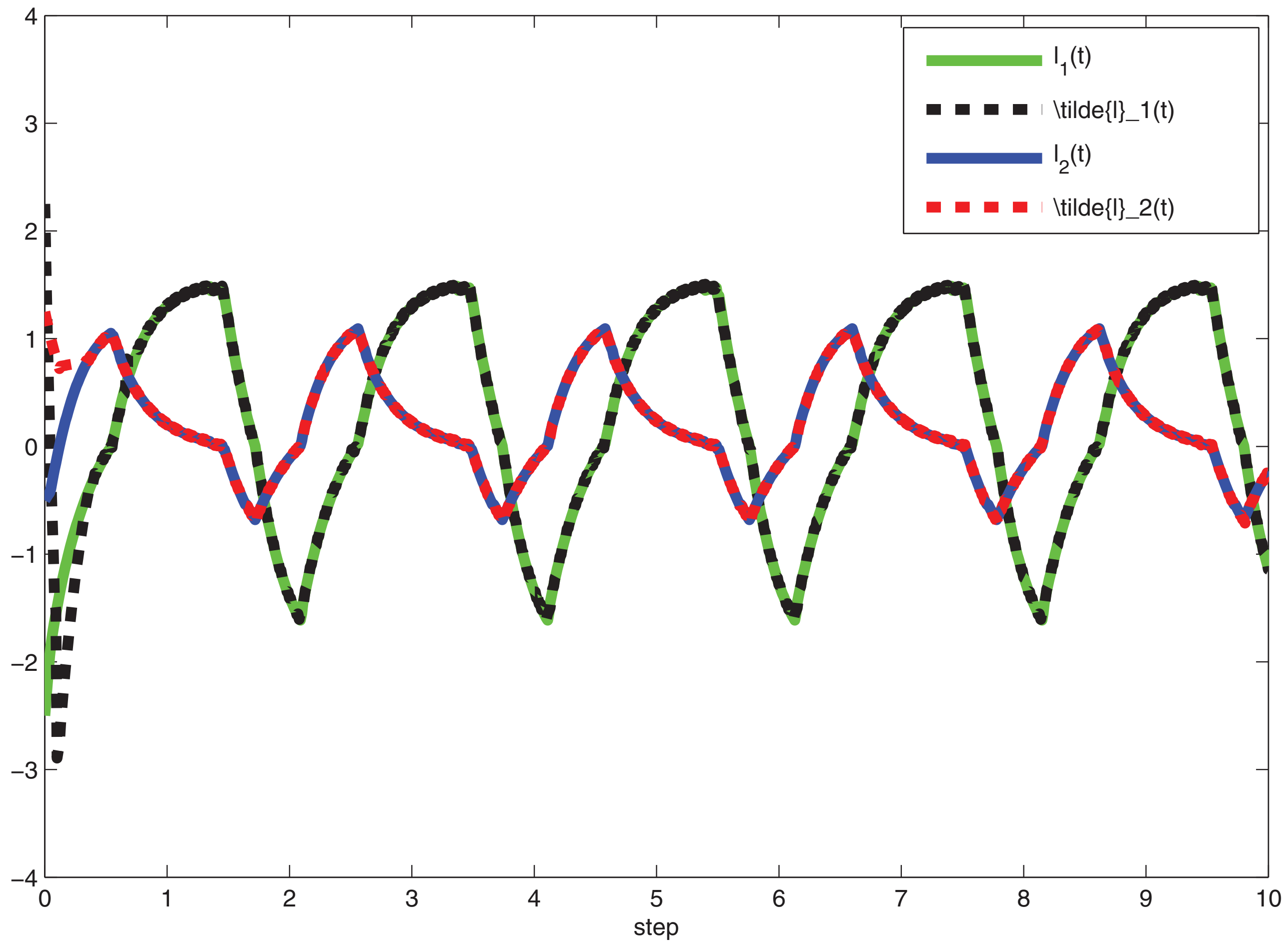

Fig. 2. The state trajectories of $l_j(t)$ vs. $\tilde{l}_j(t)$ for $j = 1, 2$.

$$y(t) = \Psi(0) + D^{-\beta}\Big[-\mathcal{B}y(t) + \mathcal{K}\mathcal{F}(y(t)) + \mathcal{S}\mathcal{F}(y(t-\eta(t))) + \mathcal{N}\Big]$$

$$= \Psi(0) - \mathcal{B}D^{-\beta}y(t) + \mathcal{K}D^{-\beta}\mathcal{F}(y(t)) + \mathcal{S}D^{-\beta}\mathcal{F}(y(t-\eta(t))) + D^{-\beta}\mathcal{N} \tag{42}$$

From assumption $[\mathcal{A}_3]$ and it is notice that

$$\|\mathcal{F}(y(t))\|_1 = \sum_{p=1}^{n+m} |\mathcal{F}(y_p(t))|$$

$$\leq \sum_{i=1}^{n}\Big\{\tilde{\mathcal{R}}_i^l\big|l_i(t)\big| + \tilde{\pi}_i^l\Big\} + \sum_{j=1}^{m}\Big\{\tilde{\mathcal{R}}_j^k\big|k_i(t)\big| + \tilde{\pi}_j^k\Big\}$$

$$\leq \Phi\|y(t)\| + \Upsilon, \tag{43}$$

where $\Phi = \max_{i\in\mathfrak{I}_n, j\in\mathfrak{I}_m}\Big\{\tilde{\mathcal{R}}_i^l,\ \tilde{\mathcal{R}}_j^k\Big\}$ and $\Upsilon = \sum_{i=1}^{n}\tilde{\pi}_i^l + \sum_{j=1}^{m}\tilde{\pi}_j^k$.

From (42) and (43), we have

$$\|y(t)\| \leq \|\Psi(0)\| + \|\mathcal{B}\|D^{-\beta}\Big[\|y(t)\|\Big] + \|\mathcal{K}\|D^{-\beta}\Big[\|\mathcal{F}(y(t))\|\Big]$$

$$+\|\mathcal{S}\|D^{-\beta}\Big[\|\mathcal{F}(y(t-\eta(t)))\|\Big] + D^{-\beta}\Big[\|\mathcal{N}\|\Big]$$

$$\leq \|\Psi(0)\| + \|\mathcal{B}\|D^{-\beta}\Big[\|y(t)\|\Big] + \|\mathcal{K}\|D^{-\beta}\Big[\Phi\|y(t)\| + \Upsilon\Big]$$

$$+\|\mathcal{S}\|D^{-\beta}\Big[\Phi\|y(t-\eta(t))\| + \Upsilon\Big] + D^{-\beta}\Big[\|\mathcal{N}\|\Big]$$

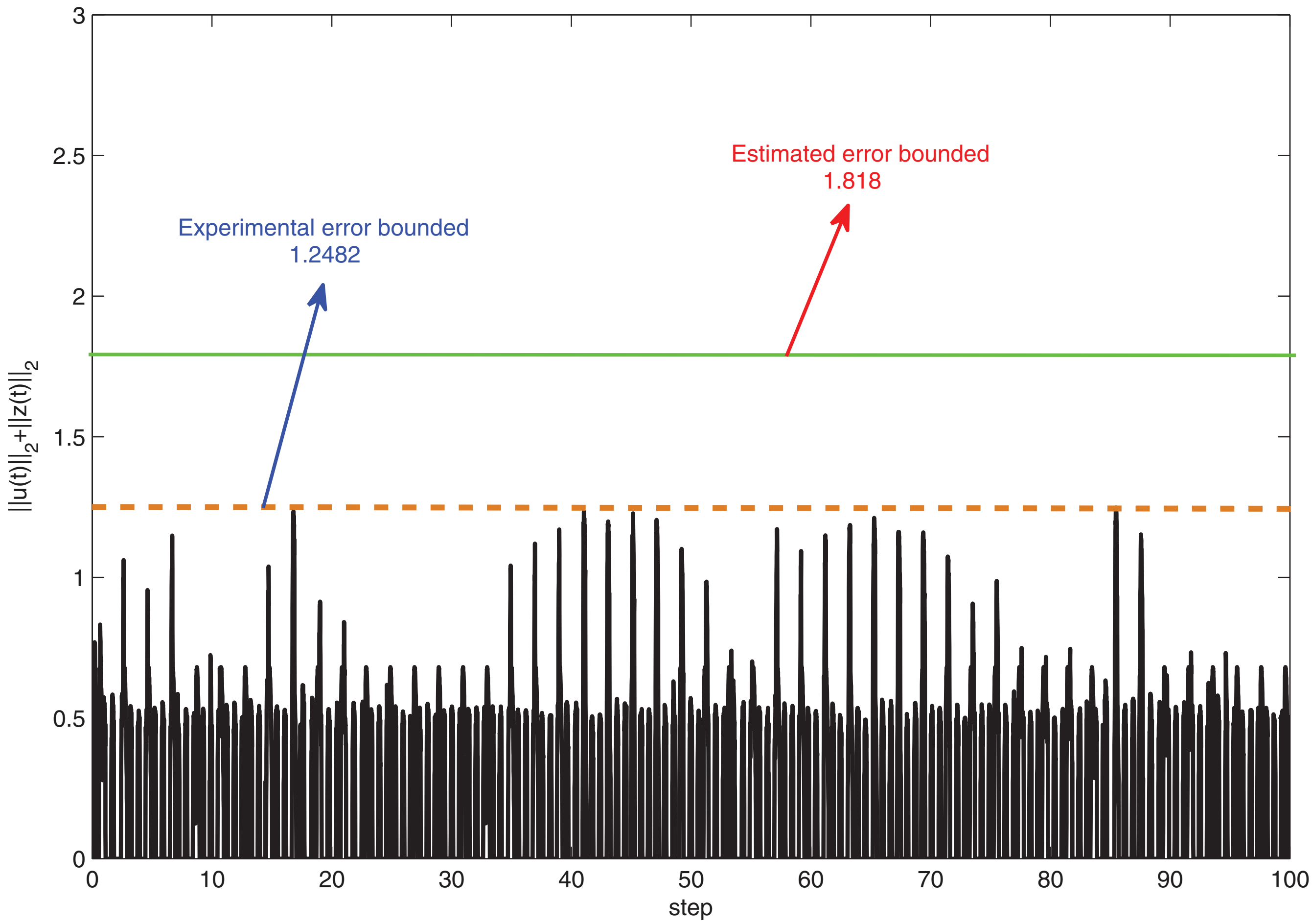

Fig. 3. Time response of controlled quasi-pinning synchronization error curves $\|u(t)\|_2 + \|z(t)\|_2$.

$$\leq \|\Psi(0)\| + \Big[\|\mathcal{B}\| + \|\mathcal{K}\|\Phi\Big]D^{-\beta}\Big[\|y(t)\|\Big] + \|\Phi\|\|\mathcal{S}\|D^{-\beta}\Big[\|y(t-\eta(t))\|\Big]$$

$$+D^{-\beta}\Big[\|\mathcal{K}\|\Upsilon + \|\mathcal{S}\|\Upsilon + \|\mathcal{N}\|\Big]. \tag{44}$$

If $t \in \Big[0, \eta(t)\Big]$, then

$$\begin{aligned}
D^{-\beta}\Big[\|y(t-\eta(t))\|\Big] &= \frac{1}{\Gamma(\beta)}\int_0^t \Big(t-\omega\Big)^{\beta-1}\|y(\omega-\eta(\omega))\|d\omega \\
&\leq \frac{1}{\Gamma(\beta)}\int_{-\eta(0)}^{t-\eta(t)} \Big(t-\kappa-\eta(\omega)\Big)^{\beta-1}\frac{\|y(\kappa)\|}{1-\eta'(\omega)}d\kappa \\
&\leq \frac{1}{\Gamma(\beta)\big(1-\hat{\eta}\big)}\int_{-\eta(0)}^{t-\eta(t)} \Big(t-\kappa-\eta(\omega)\Big)^{\beta-1}\|y(\kappa)\|d\kappa \\
&= \frac{\|\Psi\|}{\beta\Gamma(\beta)\big(1-\hat{\eta}\big)}\Big[t+\eta(0)-\eta(t)\Big]^{\beta} \\
&\leq \frac{\|\Psi\|}{\Gamma(1+\beta)\big(1-\hat{\eta}\big)}\big[2\eta\big]^{\beta}.
\end{aligned} \tag{45}$$

If $t \in \Big[\eta(t), +\infty\Big)$, we obtain

$$D^{-\beta}\Big[\|y(t-\eta(t))\|\Big] = \frac{1}{\Gamma(\beta)}\int_0^t \Big(t-\omega\Big)^{\beta-1}\|y(\omega-\eta(\omega))\|d\omega$$

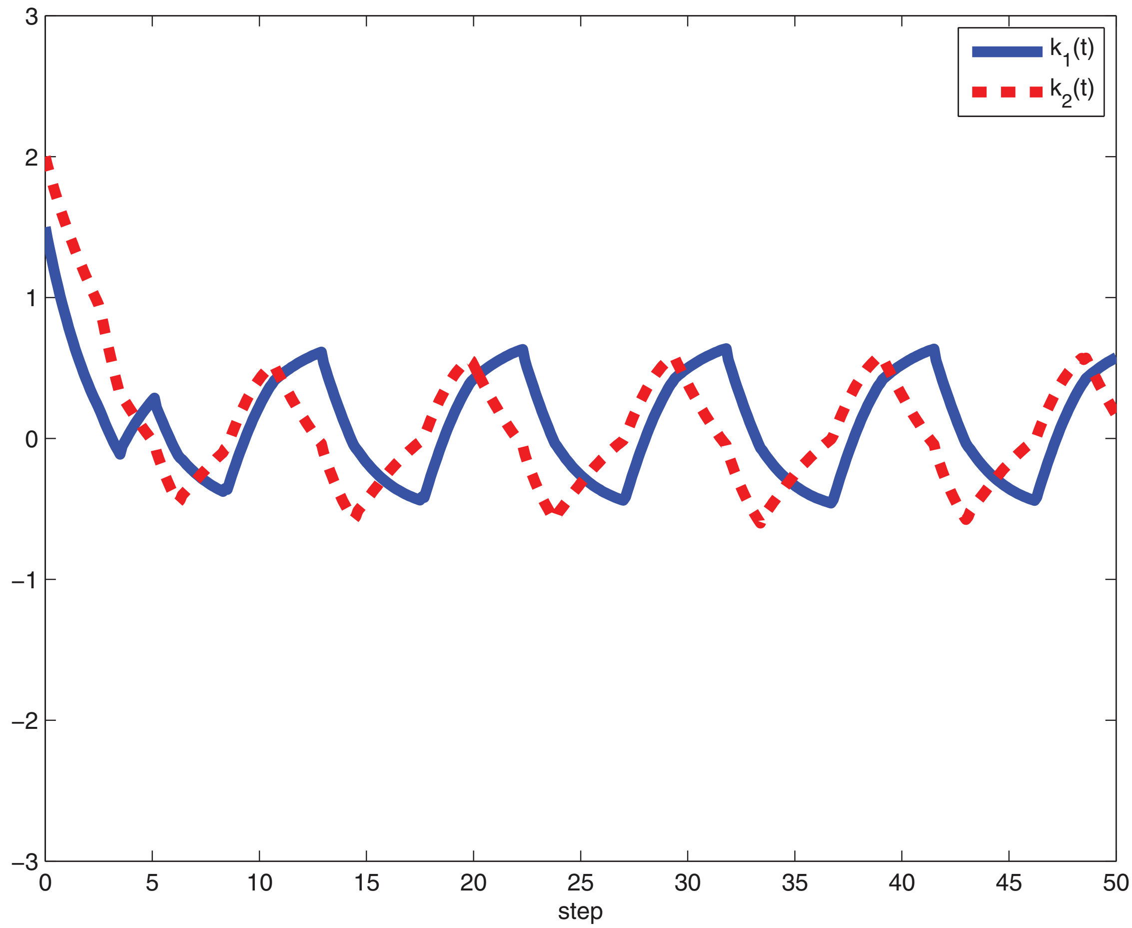

Fig. 4. The state trajectories of $k_1(t)$ and $k_2(t)$ without control inputs.

$$= \frac{1}{\Gamma(\beta)} \int_0^{\eta(t)} \Big(t-\omega\Big)^{\beta-1} \|y(\omega-\eta(\omega))\| d\omega$$

$$+\frac{1}{\Gamma(\beta)} \int_{\eta(t)}^{t} \Big(t-\omega\Big)^{\beta-1} \|y(\omega-\eta(\omega))\| d\omega$$

$$\leq \frac{1}{\Gamma(\beta)} \int_{-\eta(0)}^{\eta(t)-\eta(\eta(t))} \Big(t-\kappa-\eta(t)\Big)^{\beta-1} \frac{\|y(\kappa)\|}{1-\eta'(\omega)} d\omega$$

$$+\frac{1}{\Gamma(\beta)} \int_{\eta(t)-\eta(\eta(t))}^{t-\eta(t)} \Big(t-\kappa-\eta(t)\Big)^{\beta-1} \frac{\|y(\kappa)\|}{1-\eta'(\omega)} d\omega$$

$$\leq \frac{1}{\Gamma(\beta)\big(1-\hat{\eta}\big)} \int_{-\eta(0)}^{0} \Big(t-\kappa-\eta(t)\Big)^{\beta-1} \|y(\kappa)\| d\kappa$$

$$+\frac{1}{\Gamma(\beta)\big(1-\hat{\eta}\big)} \int_{0}^{t-\eta(t)} \Big(t-\kappa-\eta(t)\Big)^{\beta-1} \|y(\kappa)\| d\kappa$$

$$\leq \frac{1}{\Gamma(\beta)\big(1-\hat{\eta}\big)} \int_{-\eta(0)}^{0} \Big(-\kappa\Big)^{\beta-1} \|y(\kappa)\| d\kappa$$

$$+\frac{1}{\Gamma(\beta)\big(1-\hat{\eta}\big)} \int_{0}^{\sigma} \Big(\sigma-\kappa\Big)^{\beta-1} \|y(\kappa)\| d\kappa$$

$$\leq \frac{\|\Psi\|\eta^{\beta}}{\Gamma(\beta+1)\big(1-\hat{\eta}\big)} + \frac{1}{\Gamma(\beta)\big(1-\hat{\eta}\big)} \int_{0}^{t} \Big(t-\kappa\Big)^{\beta-1} \|y(\kappa)\| d\kappa. \tag{46}$$

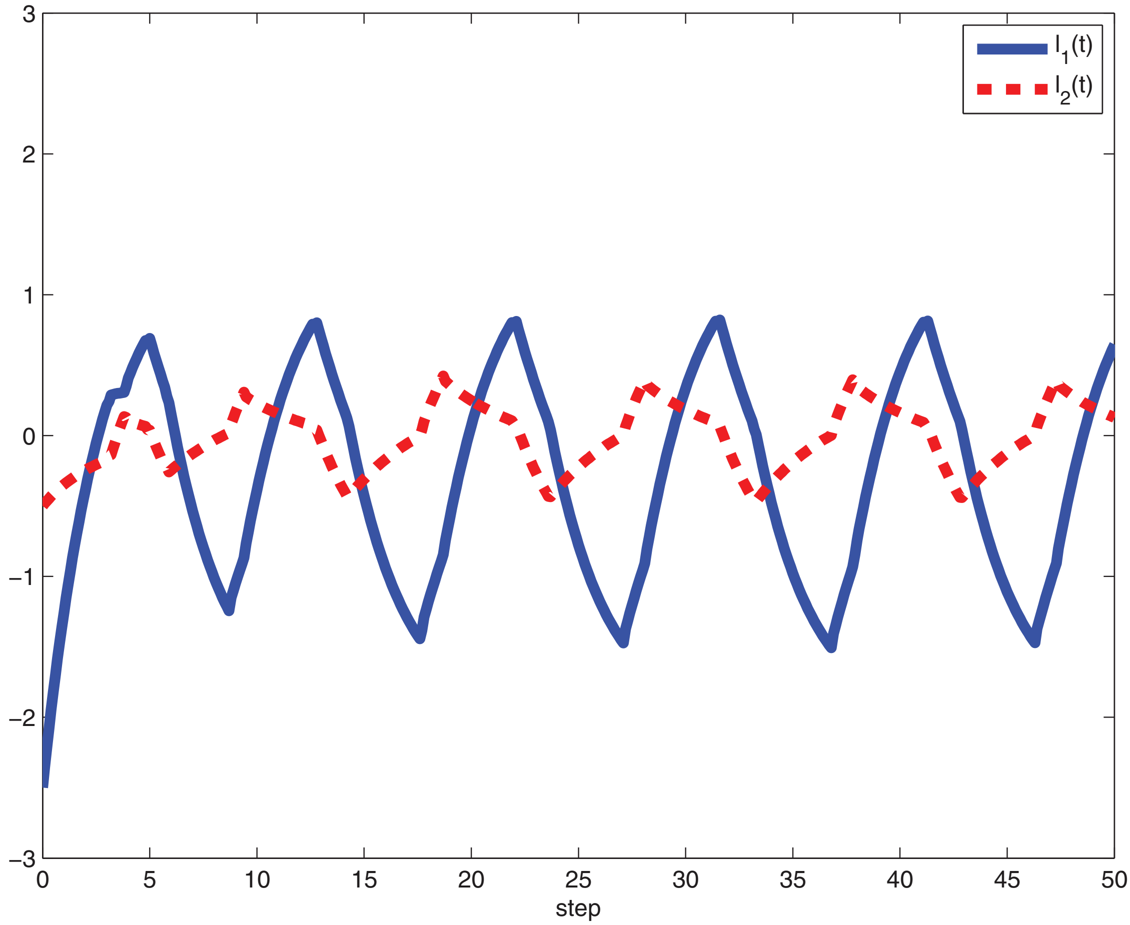

Fig. 5. The state trajectories of $l_1(t)$ and $l_2(t)$ without control inputs.

So together with (45) and (46), we obtain

$$D^{-\beta}\Big[\|y(t-\eta(t))\|\Big] \leq \frac{\|\Psi\|}{\Gamma(1+\beta)\big(1-\hat{\eta}\big)}\big[2\eta\big]^{\beta} + \frac{1}{1-\hat{\eta}}D^{-\beta}\Big[\|y(t)\|\Big] \tag{47}$$

From (44) and (47), we have

$$\begin{aligned}\|y(t)\| \leq\ & \|\Psi(0)\| + \Big[\|\mathcal{B}\| + \|\mathcal{K}\|\Phi\Big]D^{-\beta}\Big[\|y(t)\|\Big] + \|\Phi\|\|\mathcal{S}\|\Big[\frac{\|\Psi\|}{\Gamma(1+\beta)\big(1-\hat{\eta}\big)}\big[2\eta\big]^{\beta} + \frac{1}{1-\hat{\eta}}D^{-\beta}\Big[\|y(t)\|\Big]\Big] \\ & + D^{-\beta}\Big[\|\mathcal{K}\|\Upsilon + \|\mathcal{S}\|\Upsilon + \|\mathcal{N}\|\Big] \\ =\ & \|\Psi(0)\| + \Big[\|\mathcal{B}\| + \|\mathcal{K}\|\Phi + \frac{\|\Phi\|\|\mathcal{S}\|}{1-\hat{\eta}}\Big]D^{-\beta}\Big[\|y(t)\|\Big] + \frac{\|\Phi\|\|\mathcal{S}\|\|\Psi\|}{\Gamma(1+\beta)\big(1-\hat{\eta}\big)}\big[2\eta\big]^{\beta} \\ & + D^{-\beta}\Big[\|\mathcal{K}\|\Upsilon + \|\mathcal{S}\|\Upsilon + \|\mathcal{N}\|\Big] \\ =\ & \mathcal{P}D^{-\beta}\Big[\|y(t)\|\Big] + \mathcal{P}_0,\end{aligned} \tag{48}$$

where

$$\mathcal{P} = \|\mathcal{B}\| + \|\mathcal{K}\|\Phi + \frac{\|\Phi\|\|\mathcal{S}\|}{1-\hat{\eta}}$$

$$\mathcal{P}_0 = \|\Psi(0)\| + \frac{\|\Phi\|\|\mathcal{S}\|\|\Psi\|}{\Gamma(1+\beta)\big(1-\hat{\eta}\big)}\big[2\eta\big]^{\beta} + D^{-\beta}\Big[\|\mathcal{K}\|\Upsilon + \|\mathcal{S}\|\Upsilon + \|\mathcal{N}\|\Big].$$

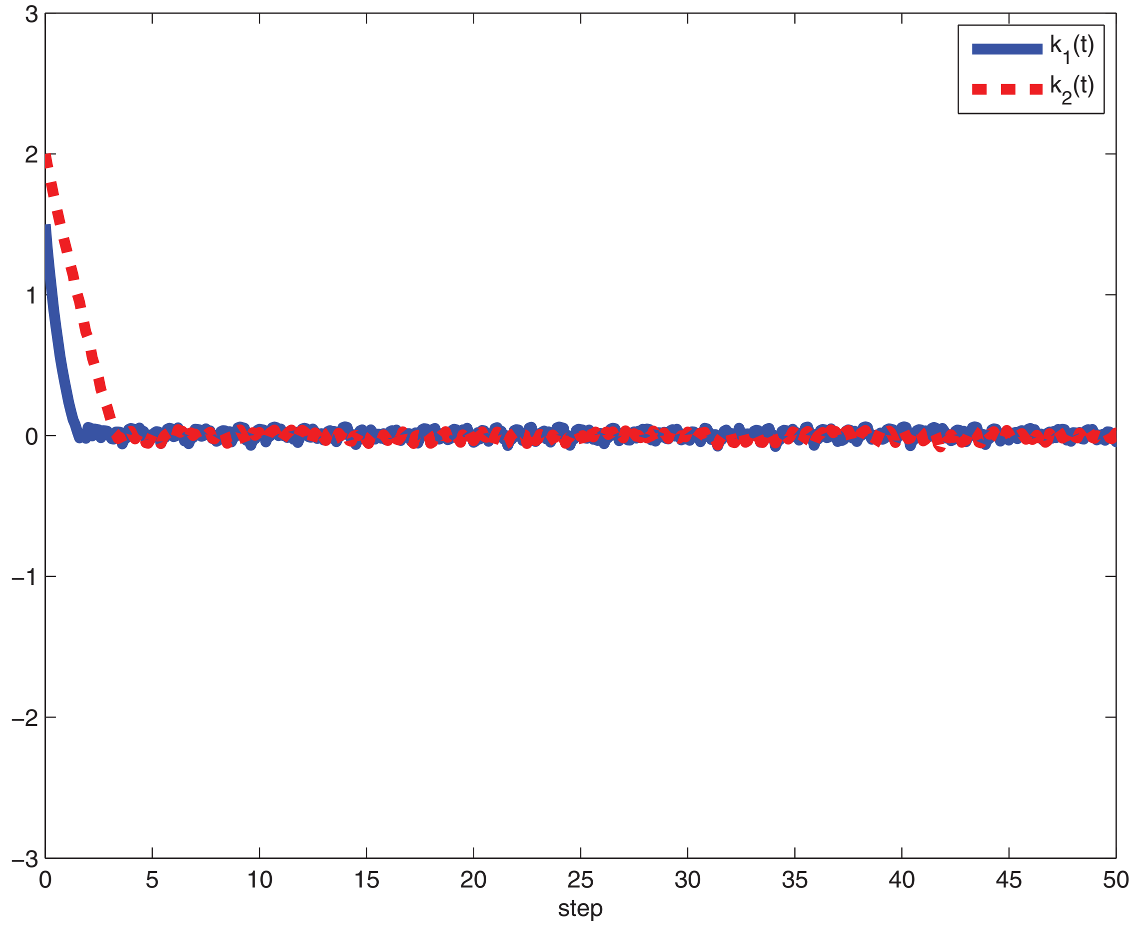

Fig. 6. The state trajectories of $k_1(t)$ and $k_2(t)$ with control inputs.

Based on Lemma 2.12, and from (48), we obtain

$$\|y(t)\| \le \mathcal{P}_0\mathcal{E}_\beta\Big[-\mathcal{P}t^\beta\Big].$$

Therefore the solution $y(t)$ of system (41) is bounded on $[0, +\infty)$, which means the solutions are exists in the Filippov sense. □

6. Numerical simulations

In this part numerical example with simulations are provided to show the effectiveness of designing pinning control method and developed theoretical results.

Example 6.1. Consider two-dimensional FBAMNNs described by the following expression:

$$\begin{cases} D^\beta k(t) = -Ak(t) + Vh(l(t)) + Wh(l(t-\eta(t))) + J \\ D^\beta l(t) = -Bl(t) + Pg(k(t)) + Qg(k(t-\eta(t))) + I \end{cases} \tag{49}$$

where $\beta = 0.999,\ k = (k_1, k_2)^T,\ l = (l_1, l_2)^T,\ I = J = (0, 0)^T,\ A = diag(7, 7),\ B = diag(7, 7), \eta(t) = 0.8 * e^t$ and

$$V = \begin{bmatrix} 0.5 & 5.65 \\ -5.2 & 0.6 \end{bmatrix},\quad W = \begin{bmatrix} -1.5 & -0.1 \\ 1 & 2.42 \end{bmatrix}\quad P = \begin{bmatrix} 2 & 4.1 \\ -1 & 1.7 \end{bmatrix},\quad Q = \begin{bmatrix} 2 & 4 \\ -3.5 & 2 \end{bmatrix}.$$

In addition the discontinuous neurons activation are $g(k) = (g_1(k_1), g_2(k_2))^T$ with $g(k_i) = 0.5 * tanh(k_i) + (sgn(k_i))$ and $h(l) = (h_1(l_1), h_2(l_2))^T$ with $h(l_j) = 0.5 * tanh(l_j) + (sgn(l_j))$, where $i = j = 1, 2$.

Obviously, the above neuron discontinuous activation functions are satisfy the assumptions $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$ with $\mathcal{R}^l_1 = \mathcal{R}^l_2 = \mathcal{R}^k_1 = \mathcal{R}^k_2 = 0.5$ and $\pi^l_1 = \pi^l_2 = 0.25,\ \pi^k_1 = \pi^k_2 = 0.4$.

The controlled response system is given by:

$$\begin{cases} D^\beta \tilde{k}(t) = -A\tilde{k}(t) + Vh(\tilde{l}(t)) + Wh(\tilde{l}(t-\eta(t))) + J + L(t) \\ D^\beta \tilde{l}(t) = -B\tilde{l}(t) + Pg(\tilde{k}(t)) + Qg(\tilde{k}(t-\eta(t))) + I + N(t). \end{cases} \tag{50}$$

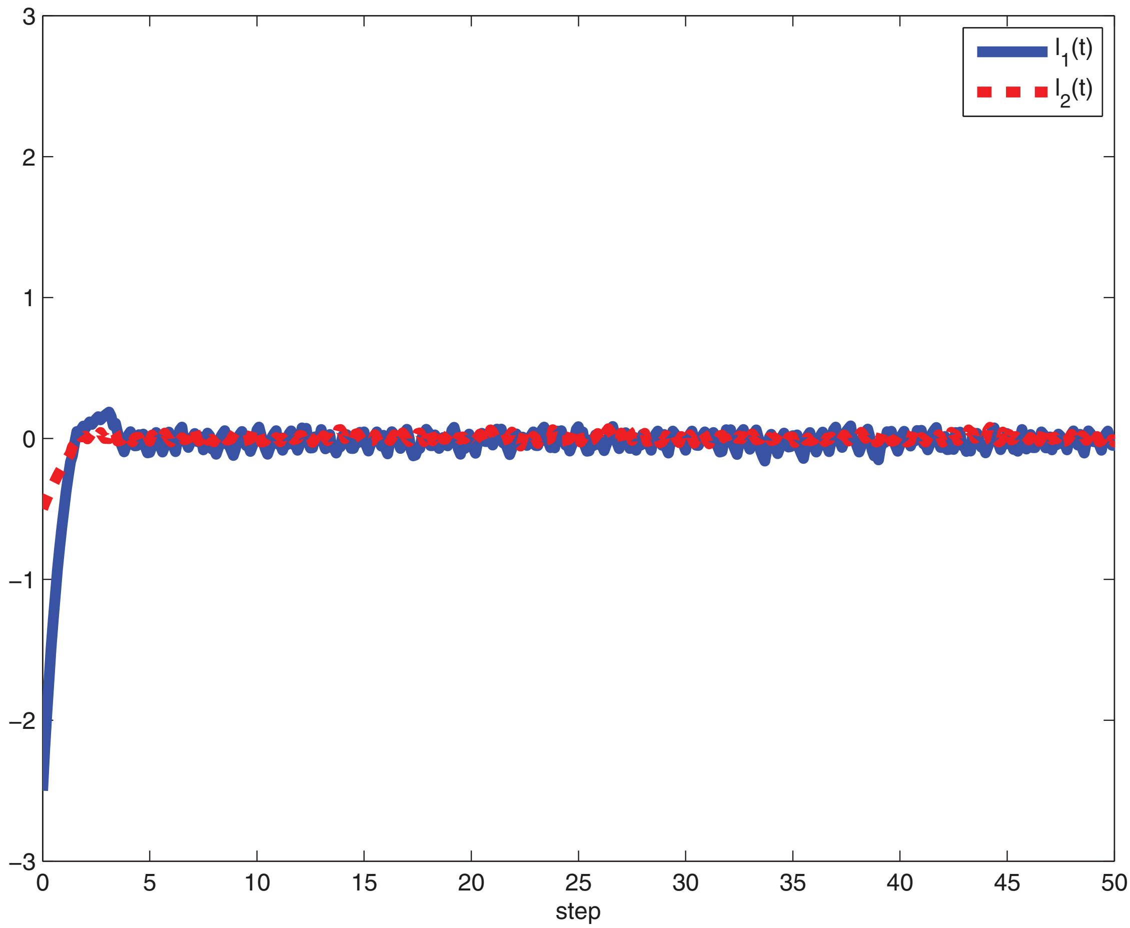

Fig. 7. The state trajectories of $l_1(t)$ and $l_2(t)$ with control inputs.

which shares the similar parameters as (35). In this example, we select first neuron in the two layers are directly controlled,i.e., $\hat{s} = \check{s} = 1$. In the following simulation, the initial values are selected as $k(\omega) = \big(k_1(\omega), k_2(\omega)\big)^T = \big(1.5, 2\big)^T$, $l(\omega) = \big(l_1(\omega), l_2(\omega)\big)^T = \big(-2.5, -0.5\big)^T$ and $\tilde{k}(\omega) = \big(\tilde{k}_1(\omega), \tilde{k}_2(\omega)\big)^T = \big(-2.75, 2\big)^T$, $\tilde{l}(\omega) = \big(\tilde{l}_1(\omega), \tilde{l}_2(\omega)\big)^T = \big(2.25, 1.25\big)^T$ for $\omega \in [-1, 0]$.

Firstly, we choose $\xi = \varepsilon = 10$ in pinning controller (11), the condition of Theorem 3.2 is satisfied. In afterwards, it is easy to get $\mathcal{H} = 25.715$, $\mathcal{L} = 3.015$ and $\Phi = 18.76$ with $\varsigma_1 = 1$, $\varsigma_2 = 0.5$, $\varsigma_3 = 0.5$, $\zeta_1 = 0.5$, $\zeta_2 = 1$ and $\zeta_3 = 0.5$. By virtue of Theorem 3.2, drive system (1) and controlled response system (6) can be realize quasi-synchronized via pinning controller (11) with estimated error bound $\sqrt{\frac{4\Phi}{\mathcal{H}-\mathcal{L}}} \approx 1.818$, which is shown in Figs. 1–3.

In Figs. 1 and 2 depict the two states evolution curves of drive-response FBAMNNs under pinning control. The controlled synchronization error curves are displayed in Fig. 3 and the obtained experimental error bound is $\|u(t)\|_2 + \|z(t)\|_2$=1.2482. From those Fig. 3, one can observe that experimental error bound is less than the theoretical error bound, which confirm the effectiveness of proposed pinning control for quasi-synchronization. This implies that, the drive-response system achieves quasi-synchronization under pinning control.

In Figs. 4 and 5 display the time response of state variables $k_1(t), k_2(t), l_1(t)$ and $l_2(t)$ with similar initial values. From those figures, the equilibrium point of FBAMNNs (49) is not stable. Manifestly, $k^* = \big(0, 0\big)$ and $l^* = \big(0, 0\big)$ is an equilibrium point of FBAMNNs (49). Based on obtaining results in Theorem 4.2, and the controller (27) with $\hat{s} = \check{s} = 1$ is transformed into the following expression:

$$\begin{cases} E_1(t) = -10sgn\{k_1(t)\} \times \left[\frac{\sum_{i=1}^{2} |k_i(t)|}{|k_1(t)|}\right] \times \sum_{j=1}^{2} |k_j(t)| - 4sgn\{k_1(t)\}, \\ F_1(t) = -10sgn\{l_1(t)\} \times \left[\frac{\sum_{j=1}^{2} |l_j(t)|}{|l_1(t)|}\right] \times \sum_{i=1}^{2} |l_i(t)| - 3.5sgn\{l_1(t)\}, \\ E_2(t) = -4sgn\{l_2(t)\},\ F_2(t) = -3.5sgn\{k_2(t)\}. \end{cases} \tag{51}$$

Now, we are able to obtain the controlled FBANNs (49) is

$$\begin{cases} D^\beta k(t) = -Ak(t) + Vh(l(t)) + Wh(l(t-\eta(t))) + J + E(t) \\ D^\beta l(t) = -Bl(t) + Pg(k(t)) + Qg(k(t-\eta(t))) + I + F(t), \end{cases} \tag{52}$$

where $E(t) = \big(E_1(t), E_2(t)\big)^T$ and $F(t) = \big(F_1(t), F_2(t)\big)^T$ are defined in (51) and other parameters are similar as in (49). Taking $\varsigma = \zeta = 3,\ \delta = 1,\ \mu = 1.5$. By simple manipulation, we can get

$$4 = \tau > \max_{i\in\Im_n}\left\{\sum_{j=1}^{m}\Big(|v_{ij}| + |w_{ij}|\Big)\pi_j^k\right\} = 3.688 \text{ and}$$

$$3.5 = \theta > \max_{j\in\Im_m}\left\{\sum_{i=1}^{n}\Big(|p_{ji}| + |q_{ji}|\Big)\pi_i^l\right\} = 3.025.$$

Moreover, via LMI MATLAB toolbox, we find that linear matrix inequality is viable and the feasible solution is as follows:

$$\Lambda = \begin{bmatrix} 3.6993 & 0 \\ 0 & 3.7531 \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} 3.7476 & 0 \\ 0 & 3.7147 \end{bmatrix},$$

$$\Phi_1 = \begin{bmatrix} 41.8784 & 0 \\ 0 & 40.1113 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} 10.5702 & 0 \\ 0 & 10.6853 \end{bmatrix}, \quad \Psi_1 = \begin{bmatrix} 51.7706 & 0 \\ 0 & 47.7329 \end{bmatrix},$$

$$\Psi_2 = \begin{bmatrix} 8.4258 & 0 \\ 0 & 8.7657 \end{bmatrix}.$$

In simulation part, the state trajectories of $k_1(t), k_2(t), l_1(t)$ and $l_2(t)$ of system (49) are displayed in Figs. 6 and 7 with control inputs (51). From those figures the sate trajectories of (52) converges to origin. That is, the system (49) can be β-exponential stabilized via pinning controller (27).

7. Conclusion

In this paper, the quasi-pinning synchronization and β-exponential stabilization has been studied for fractional order BAM neural networks with time-varying delays and limiting discontinuous neuron activations based on pinning control policy. Via the framework Filippov's theory and set valued map analysis, the quasi-synchronization issue for a class of drive-response discontinuous right hand side of the FBAMNNs dynamical system is formulated. Furthermore, a novel pinning control policy was designed to guarantee the quasi-synchronization for considering drive-response discontinuous FBAMNNs error dynamical system. Then, based on Kakutani's fixed point theorem of set-valued map analysis, the global existence and equilibrium point of discontinuous right hand sides of FBAMNNs is investigated, while a new brand of some algebraic β-exponential stabilization criteria of such FBAMNNs system is displayed by a designing nonlinear pinning controller. A numerical computer simulations is also presented to demonstrate the effectiveness of the pinning control method. Moreover, the synchronization problem has played an important role in engineering applications, such as information sciences [58] and secure communication [59–61]. Pinning control techniques can be suitable for various glorious dynamics such as FNNs [18], memristor based FNNs [25], memristor based complex-valued FNNs [57], and fractional order T-S fuzzy neural networks [62]. This will appear in soon.

Declaration of Competing Interest

We confirm that we no conflict of interest.

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