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Optimizing inventory decisions for a closed–loop supply chain model under a carbon tax regulatory mechanism

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ABSTRACT

Green supply chain management encompasses every level of the supply chain. The core green supply chain management strategies include closed–loop manufacturing and reduction of carbon footprint. These strategies enable firms to improve their environmental profile and to comply with environmental regulations. This paper deals with a supply chain system integrating manufacturing, remanufacturing and repair activities (closed–loop manufacturing) to face a time-varying demand under the regulatory framework for carbon tax. Initially, the total cost of the system is provided and then a mixed integer nonlinear programming problem is formulated aiming to determine the optimal policy i.e. the manufacturing, remanufacturing and repairing cycles. After decomposing the original problem into two pure manufacturing and remanufacturing sub-problems, the existence of their optimal solution is proved and then a simple method, which relies on a finite search scheme, is used to determine the overall optimal solution. Through a variety of numerical examples and a sensitivity analysis, the effect of the different system parameters on costs and environmental efficiency are provided, such as: returns, carbon emissions generated per activity and rates of any activity. The main result of this analysis indicates that the proposed model is fairly robust to the parameters' changes, however the tax on carbon emissions has a serious impact on the system optimal cost.

1. Introduction

Nowadays, many companies are actively integrating sustainability principles as they are associated with corporate social responsibility. At the same time, a growing number of consumers require practices that are oriented towards environmentalism and companies are striving to address their concerns by improving internal operations. The most effective measures at sustainability are those dealing with a reconfiguration of the supply chain. The various stages of supply chain offer plenty of flexibility in incorporating eco-friendly practices. The positive public image and reputation from identifying and implementing sustainable supply chain practices can yield numerous benefits for firms and organizations (Suzanne et al., 2020). In this context, companies are required to redefine their supply chain management tools under an economic and environmental point of view by altering operational policies such as manufacturing, transport, inventory, etc. (Benjaafar et al., 2013). Thus, suitable decision-making methods are building that balance of environmental efficiency against costs and so the ordinary supply chain is transformed into a green supply chain. Green supply chain management can resolve many issues surrounding commercial manufacturing techniques. The core green supply chain strategies include, among others, closed–loop manufacturing and reduction of carbon emissions (Min and Kim, 2012).

Human activities are responsible for almost all of the increase in greenhouse gas emissions in the atmosphere over the last 150 years. Excessive carbon dioxide emission is regarded as a key contributor to global warming and hence a major threat to the environment. The European Commission (EU), through the European Green Deal roadmap (https://ec.europa.eu), aims to make the EU's economy sustainable with no net greenhouse gas emissions by 2050, thus turning the climate and environmental challenges into opportunities by making the transition just and inclusive for all. Specifically, the plan of low–carbon economy indicates that by 2050, emissions must be reduced by 80% below 1990 levels. For this purpose, emissions must be reduced by 40% until 2030 and by 60% until 2040. In order to mitigate carbon emissions, governments and businesses are looking for ways to minimize operating CO_2

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emissions to limit the supply chains' carbon footprints (Shu et al., 2018). In the literature, the key policies that are suggested to reduce carbon emissions are carbon tax and cap and trade (Benjaafar et al. (2013); Zakeri et al. (2015); Fareeduddin et al. (2015); Haites (2018); Xu et al. (2019)). Cap and trade specify the level of permissible emissions, while simultaneously allowing the market to determine the cost of reducing emissions to that level. Carbon tax is the obverse of cap and trade: it defines its price rather than setting the level of permitted emissions. Companies covered by the cap would balance the cost of lowering their emissions against the tax they would pay if they continued to emit at their current level. Carl and Fedor (2016) provides a survey that presents the use of public revenues generated by carbon taxes as well as cap and trade policies.

End-of-life product recovery and repair of defective items constitute, also, strategic choices that comply with the environmental challenges that influence the image of the companies (Reimann et al., 2019). Remanufaturing and rework procedures are an important part of manufacturing logistics having several connected goals that help support the logistics process and ultimately lead to better manufacturing practices. Recovery reasons involve not only some strict laws passed by governments but cost reduction of raw materials and cost reduction of waste disposal. For example, according to Zhou and Sun (2019) if 95% of mobile phones are recycled and remanufactured, then one billion euros in raw materials per year will be saved. A reduction of materials input, through repair, re-use and recycling, requires less energy, reducing pollution and carbon emissions. Product recovery includes decisions such as, remanufacturing, repairing and finally disposing of some used goods. So, in such systems, the options to meet the customers demand could be: (1) by externally ordering/producing new items, (2) by repairing defective items and (3) by recovering used ones (Fleischmann et al. (1997); Srivastava (2008); Dekker et al. (2013); Govindan et al. (2015); Sonntag and Kiesmüller (2018); Yu et al. (2019); Nahr et al. (2020)). A distinctive paradigm is that of Lexmark, which has a market share of around 10% in laser printer cartridges, and all laser cartridges were imported from China and Mexico in 2012 and none were produced in Europe. Lexmark decided to remanufacture these cartridges close to the sources of returned cartridges, in order to avoid these long-distance shipping costs and thus there was a chance to produce again in Europe and to build new employment (https://ec.europa.eu).

In the present paper, a continuous review production-inventory model with finite planning horizon is studied, under the joint consideration of remanufacturing, repairing and carbon tax, as sustainability options. For such a problem the planning horizon is made by a period of regular manufacturing (with a possibility of repair) followed by a period of remanufacturing (with a possibility of repair). Additionally, each of these periods is made up of cycles of the corresponding manufacturing or remanufacturing activity. Timings for regular manufacturing and remanufacturing cycles need to be found along with the frequencies of these activities. The decision problem of devising an optimal inventory policy is complex due to the various parameters involved. Moreover, the decision problem leads to the problem of solving a mixed integer nonlinear programming problem (MINLP) of which the solution is not readily available. This paper aims at addressing the following questions pertaining to the model:

1. Since the decision problem requires to set up a MINLP with its main inputs, inter-alia, the costs of manufacturing, repairing, remanufacturing, holding and carbon emission; then what is a suitable formulation for the problem of determining an appropriate production-inventory policy within the carbon tax framework?
2. What is the optimal policy over the entire planning horizon (i.e. what is the manufacturing, remanufacturing and repairing quantities that minimize the total cost) and how robust is this policy to changes in the model's parameters?
3. What are the conditions that ensure the existence and the uniqueness of the problem's optimal solution?
4. What is the impact of demand trend on the cycle length and what is the sustainability factor with the highest influence on the system's cost operation?

Benkherouf et al. (2014) laid the basis for the manufacturing–remanufacturing part of the proposed model, while Benkherouf and Omar (2017) provided insight for the repair part of the model. The proposed model unifies and extends both previous ones, along with a more general form of repair and with a carbon tax regulation mechanism. Carbon tax is more efficient and less costly than other ways of reducing carbon emissions (https://energynews.us/). Also our model although developed for repair that occurs after the completion of a regular manufacturing–remanufacturing cycle, it can equally handle the repair scheme of Benkherouf and Omar (2017). The resulting model therefore provides an answer to question 1 above and at the same time broadens its applicability by allowing the examination of a number of existing models in a unified way. Answers to the other questions will arise from the careful mathematical analysis of the MINLP. The existence and the derivation of the optimal policy will be shown to follow from the recent work of Benkherouf and Gilding (2020), while uniqueness will be provided as a consequence of results in Benkherouf and Gilding (2009).

The remainder of the paper is structured as follows: The literature review on inventory models is presented in Section 2, focusing on papers that incorporate carbon emissions and product recovery. Section 3 provides the assumptions and notations of the inventory model considered while Section 4 includes the mathematical model and the detailed cost structure. Section 5 gives the optimization technical details and the solution and Section 6 gives some numerical examples to illustrate the model, as well as some managerial insights. Finally, Section 7 provides findings and directions for future research.

2. Literature review

Over the past ten years, more emphasis has been placed on research on inventory control with considerations of carbon emissions. One sequence of studies is conducted in the context of the Economic Order Quantity (EOQ), a fundamental concept for inventory management. In this context, Bonney and Jaber (2011) introduced an expansion of the EOQ model, dubbed "Enviro-EOQ", which takes into account the expense of both disposal and transport associated emissions. Hua et al. (2011) presented an environmental inventory model under carbon cap-and-trade and derived the optimal order quantity. They also studied the impacts of carbon cap and price on the order quantity, emissions and overall cost. Under various environmental restrictions, Chen et al. (2013) investigated the optimal order quantities and concluded that a company can limit carbon emissions without significantly increasing costs. Battini et al. (2014) suggested a green EOQ model that involves three parts: (i) the warehousing carbon emission costs, on the basis of the storage space, (ii) the picking and removal of obsolete inventory, and (iii) the transportation, based on quantity and distance. They evaluate the effect of sustainability factors on ordering decisions. Hovelaque and Bironneau (2015) analyzed an economic order quantity model in which the deterministic demand rate of the product depends on its price and annual carbon emissions. Furthermore, He et al. (2015) analyzed the effect of production and regulation parameters on the optimum lot sizing and emissions, based on an EOQ-type model. In a similar direction as EOQ, Taleizadeh et al. (2018) studied the Economic Production Quantity (EPQ) model taking into account carbon emission costs and shortages. Recently, Tao and Xu (2019) examined the impact of regulatory policies and consumers' low-carbon awareness on optimum order quantity, on emission levels and on overall costs, in an EOQ-type model.

The literature on product recovery topics, especially relating to supply chain and inventory control models is extensive. We just mention four reviews, which propose new paths in reverse logistics research (Sasikumar and Kannan (2009), Agrawal et al. (2015), Govindan et al.

(2015) and Bazan et al. (2016)). Although the literature of inventory control models with product recovery options is extensive, only a few research studies investigate the impact of carbon emissions considerations on production and remanufacturing lot sizing rulings. Remanufacturing is assumed to economize production costs and to lower carbon emissions and businesses are gradually embracing it. In order to boost demand for recovered products and promote the recycling of used ones, a lot of businesses begun implementing "Trade-Old-for-Remanufactured (TOR)" programs which encourage consumers to return used items for credits to purchase remanufactured products (Han et al., 2017). Carbon taxes and remanufacturing impact manufacturers' production and operating costs and can therefore greatly affect manufacturers' inventory and production decisions, thereby promoting the reduction of carbon emissions. Bazan et al. (2015) introduced a closed-loop inventory model where they assumed emissions from manufacturing, remanufacturing and transportation activities. They determined the optimal number of production and remanufacturing batches per cycle, as well as the optimal number of times which a product can be remanufactured. Dwicahyani et al. (2017) studied a two echelon inventory control model with remanufacturing, carbon emission and energy effects. By incorporating carbon emissions cost and energy cost in the total cost function, they determined the optimal value of shipment lot size, the optimal number of shipments and the optimal number of remanufacturing generations. García-Alvarado et al. (2017) introduced an inventory control model incorporating decisions for remanufacturing and aiming to reduce carbon emissions. They examined how reforming inventory control policies with remanufacturing, in the context of a cap-and-trade mechanism, should be changed. An EOQ-type inventory model, under carbon emission and product recovery activities, was studied by Shu et al. (2017). In their paper, they found the optimal manufacturing and remanufacturing quantities in closed form. Turki et al. (2018) presented a closed-loop supply chain model with machine failures and carbon emission constraints. They determined the optimum lengths of production and remanufacturing times as well as the optimum capacity of the serviceable stocks of new and remanufactured products. Zouadi et al. (2018) suggested a dynamic lot sizing model that takes into account the consumers' return quantities with carbon emission constraints due to production, remanufacturing and transportation activities. In order to decide the optimal lot size for remanufacturing and disposal, Condeixa et al. (2020) recently proposed an EOQ-type model that comprises economic, environmental and social parameters.

In the present paper a finite planning horizon lot sizing inventory model is introduced by integrating activities of manufacturing, repairing and remanufacturing under carbon tax constraints. Specifically, this paper proposes an inventory model, which integrates the possibility of repairing items during the manufacturing and remanufacturing process. This means that detected defective items during manufacturing and remanufacturing process are repaired. Items, which have been subjected either to repair or remanufacturing, are of quality comparable to new items. Costs accrued, during the manufacturing process, include the usual holding-manufacturing-repairing-remanufacturing costs in addition to a carbon emission production tax. The carbon tax mechanism is used as a way to minimize emissions, rather than the cap-and-trade system, as some empirical models indicate that carbon tax conduces to a 5–15% decrease in carbon emissions (Murray and Rivers, 2015). In addition, carbon tax is known to be the most efficient tool for encouraging remanufacturing, which is the quickest and easiest way to curb global greenhouse gas emissions with greater social benefits and virtually no adverse effects on economic development while efficiently curbing carbon emissions (Liu et al., 2015). Under this regulatory mechanism, the aim is to determine the manufacturing, remanufacturing and repairing plan, which minimizes the total cost comprising by set up costs, production costs, inventory costs and emission taxes. This turns out to be the solution of a mixed-integer non linear programming MINLP problem.

Benkherouf and Gilding (2009) and Benkherouf and Gilding (2020) proposed a general theory for solving a class of MINLP problems that represent finite horizon, continuous review inventory systems. The theory deployed in Benkherouf and Gilding (2009) leads to, under some technical conditions, a unique optimal solution to the MINLP. These technical conditions require, in some cases, that the demand rate function belongs to a certain class of functions. Benkherouf and Gilding (2020) provided a solution to the MINLP under a very mild technical condition. This solution is not necessarily unique nevertheless it could provide a flexible course of actions for decision makers. However, the above–mentioned results were originally developed for EOQ type-inventory models and therefore are not readily applicable to inventory-production models. In general, some mathematical pre-processing is needed before the theory can be applied. Examples of previous successful applications of this pre-processing are found in Benkherouf et al. (2014) and Benkherouf and Omar (2017). Benkherouf et al. (2014) paper deals with a remanufacturing-production–inventory model where no repair is allowed over the whole planning horizon. Benkherouf and Omar (2017) studied a model that merges the formalism of Jamal et al. (2004) into the classical production–inventory model, assuming that the defective items are reworked as new products in the last stage of a regular production. The present paper integrates manufacturing, repairing, remanufacturing and carbon tax options, where a different repair scheme to that of Jamal et al. (2004) is adopted leading to a new model. According to the new repair scheme, the repair occurs at the end of the production and the remanufacturing run. Moreover, the repair rates are allowed to differ from the regular production rates. The total cost of the corresponding MINLP, for the new model, contains the usual quadratic form of Omar and Smith (2002). This form is also encountered in Benkherouf et al. (2014) and Benkherouf and Omar (2017). The solution of the MINLP corresponding to the manufacturing-repairing-remanufacturing model of this work will be solved completely. Table 1 presents a brief comparison among the proposed model and the most closely related papers to it.

3. Assumptions and notation

The following assumptions and notations are provided for describing our aforementioned system.

3.1. Assumptions

(1) The inventory system deals with a single product.
(2) The demand rate of the item is given by a strictly positive real-valued continuous function D defined on the interval $[0, H]$, $H > 0$.
(3) The manufacturing (remanufacturing), m_m (m_r), rate is constant for all $t \in [0, H]$.
(4) Items are returned according to a continuous function of time given by $c(t) = \varphi D(t)$, $0 \leq \varphi \leq 1$ (Benkherouf et al., 2014).
(5) All of the used items that collected, are remanufactured and considered as new ones (Fleischmann et al., 1997).
(6) The planning horizon is comprised of manufacturing-repairing cycles and remanufacturing-repairing cycles.
(7) During a manufacturing cycle non quality-conforming items are produced at constant rate $a_m > 0$, while in the remanufacturing cycles, they are produced at a rate ($a_r > 0$). Although an inspection process is carried out to categorize the condition of the items in manufacturing and remanufacturing cycles, we assume that the inspection cost is included in the corresponding setup costs.
(8) For manufacturing and remanufacturing rates it is assumed that $m_m > D(t) + a_m$ and $m_r > D(t) + a_r$, for every $t \in [0, H]$ (Benkherouf et al., 2014).
(9) All defective items are repaired at a fixed repair rate $p_m > 0$ during the manufacturing process and at a rate $p_r > 0$ during the remanufacturing process. Repaired and remanufactured items are brought to quality equivalent to that of new items.

Table 1
Comparison of the proposed model with the most related ones.

References	Demand rate		Production rate		Eco-conscious issues			Horizon	
	Constant	Time varying	Infinite	Finite	Remanufacturing	Repair/rework	Carbon emission	Infinite	Finite
Jamal et al. (2004)		✓		✓		✓			✓
Benkherouf and Gilding (2009)		✓	✓						✓
Hua et al. (2011)	✓		✓				✓	✓	
Chen et al. (2013)	✓		✓				✓	✓	
Battini et al. (2014)	✓		✓				✓	✓	
Benkherouf et al. (2014)		✓		✓	✓				✓
Bazan et al. (2015)	✓			✓	✓		✓	✓	
Benkherouf and Omar (2017)		✓		✓		✓			✓
Dwicahyani et al. (2017)	✓		✓		✓		✓	✓	
Shu et al. (2017)	✓			✓	✓		✓	✓	
Taleizadeh et al. (2018)	✓			✓			✓	✓	
Tao and Xu (2019)	✓						✓	✓	
Condeixa et al. (2020)	✓		✓		✓		✓	✓	
Present paper		✓		✓	✓	✓	✓		✓

(10) During the repair process no faulty products are made (Benkherouf and Omar, 2017).
(11) The demand is satisfied by newly manufacturing items, remanufacturing items, and the repaired ones.
(12) The planning horizon is made up of n_1 regular manufacturing cycles (runs) and n_2 remanufacturing cycles (runs).
(13) Under carbon tax policy, all activities, which are associated with emitting carbon, are penalized financially by a tax per unit of carbon emitted.
(14) The lead time is zero and shortages are not permitted.

3.2. Notations

$I_i(t)$	the inventory level in Location i, $i = 1, 2$.
t_{i-1}	the manufacturing starting time, in cycle i, where $i = 1, \ldots, n_1$ and $t_0 = 0$ (decision variables).
t_i^m	the manufacturing completion time, in cycle i, $i = 1, \ldots, n_1$ (decision variables).
τ_i^p	the stopping time of the repair process during i manufacturing run, $i = 1, \ldots, n_1$ (decision variables).
r_{k-1}	the remanufacturing starting time, in cycle k, $k = 1, \ldots, n_2$ with $r_0 = t_{n_1}$ (decision variables).
r_k^r	the remanufacturing stopping time, in cycle k, $k = 1, \ldots, n_2$ (decision variables).
z_k^p	the stopping time of the repair process during k remanufacturing run, $k = 1, \ldots, n_2$ (decision variables).
α	the tax charged per unit of CO_2 emitted (in \$/tone CO_2).
m_m	the manufacturing rate (units/unit time).
m_r	the remanufacturing rate (units/unit time).
p_m	the repair rate during the manufacturing process (units/unit time).
p_r	the repair rate during the remanufacturing process (units/unit time).
a_m	the rate of non quality–conforming items during the manufacturing process (units/unit time).
a_r	the rate of non quality–conforming items during the remanufacturing process (units/unit time).
c_m	the set up cost of manufacturing (in \$/setup).
c_r	the set up cost of remanufacturing (in \$/setup).
c_p	the set up cost of repair(in \$/setup).
u_f	the unit collection cost of used items (in \$/unit).
u_m	the unit manufacturing cost (in \$/unit).
u_r	the unit remanufacturing cost (in \$/unit).
u_p	the cost of repair per unit(in \$/unit).
h_i	the inventory holding cost for Location i, $i = 1, 2$ (in \$/unit/unit time).
c_m^e	the carbon emitted from set up process in a manufacturing cycle (tones CO_2).
c_r^e	the carbon emitted from set up process in a remanufacturing cycle (tones CO_2).
c_p^e	the carbon emitted from set up process in a repair cycle (tones CO_2).
u_m^e	the carbon emitted by manufacturing a new product unit (tones CO_2)/unit).
u_r^e	the carbon emitted by remanufacturing a product unit (tones CO_2)/unit).
u_p^e	the carbon emitted by repair a product unit (tones CO_2)/unit).
u_f^e	the carbon emitted by returning a product unit (tones CO_2)/unit).
h_i^e	the carbon emitted by holding a unit in Location i, $i = 1, 2$ (tones CO_2)/unit).

4. System operation and cost formulation

The system under consideration is a single item deterministic manufacturing-remanufacturing-repairing system over a finite planning horizon of length H and it is illustrated in Fig. 1.The inventory system has two locations for storing goods, labeled as L_1 and L_2. Location L_2 stores manufacturing and remanufacturing goods, while location L_1 stores exclusively returning ones. The planning horizon is made up of two phases. In the initial phase customers' demand is satisfied from new manufacturing lots of unequal size. Customers' demand in the second phase is met from remanufacturing lots of unequal size. During the manufacturing and remanufacturing runs defective items can also be produced. These items are reworked and converted into items of perfect quality, after the end of each manufacturing and remanufacturing runs where they were produced. Carbon emissions are generated by every process (manufacturing, remanufacturing, repairing, collection of used products). Each phase is made up of cycles comprising manufacturing (remanufacturing) and repairing of the items. Items produced during the cycle that are classified as defective are repaired and reinstated to stock as new products during that cycle. The first phase lasts up to some time r_0, while Phase 2 starts at r_0 and ends at H, see Fig. 2.

We shall next examine the changes in the evolution of the level of inventory $I_\ell(t)$, $\ell = 1, 2$ in locations L_1 and L_2 starting with location L_2.

4.1. Evolution of inventory in location L_2

A typical cycle, i, in Phase I, at Location 2, operating on an interval $[t_{i-1}, t_i)$. Stock is built up to time $t_i^m < t_i$ and products that found to be defective on the interval $(t_i, t_i^m]$ are repaired on the interval $[t_i^m, \tau_i^p)$, where $t_i^m < \tau_i^p < t_i$. Subsequently, the inventory level depletes, due to demand, to reach zero. If $t_i < r_0$, a new cycle is initiated, otherwise the manufacturing stage comes to its end.

It is a simple to see that the evolution of the inventory on $[t_{i-1}, t_i)$ can be described by the equations:

$$\frac{dI_2(t)}{dt} = m_m - a_m - D(t), \quad t_{i-1} \le t < t_i^m, \tag{1}$$

$$\frac{dI_2(t)}{dt} = -D(t) + p_m, \quad t_i^m \le t < \tau_i^p, \tag{2}$$

$$\frac{dI_2(t)}{dt} = -D(t), \quad \tau_i^p \le t < t_i. \tag{3}$$

with boundary conditions $I_2(t_{i-1}) = 0$, and $I_2(t_i) = 0$. In addition, the function I_2 is continuous on $[t_{i-1}, t_i]$.

Note that it is possible to assume that repair is carried out before time τ_i^p, in which case equations (1)–(3) need to be modified accordingly. This case is found in Jamal et al. (2004) and Benkherouf and Omar (2017) for

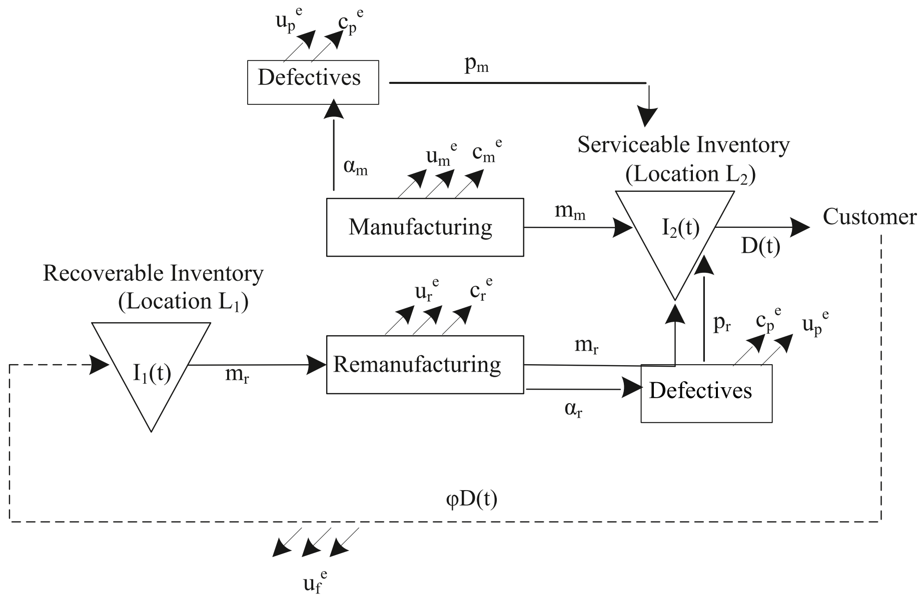

Fig. 1. Graphical representation of the system.

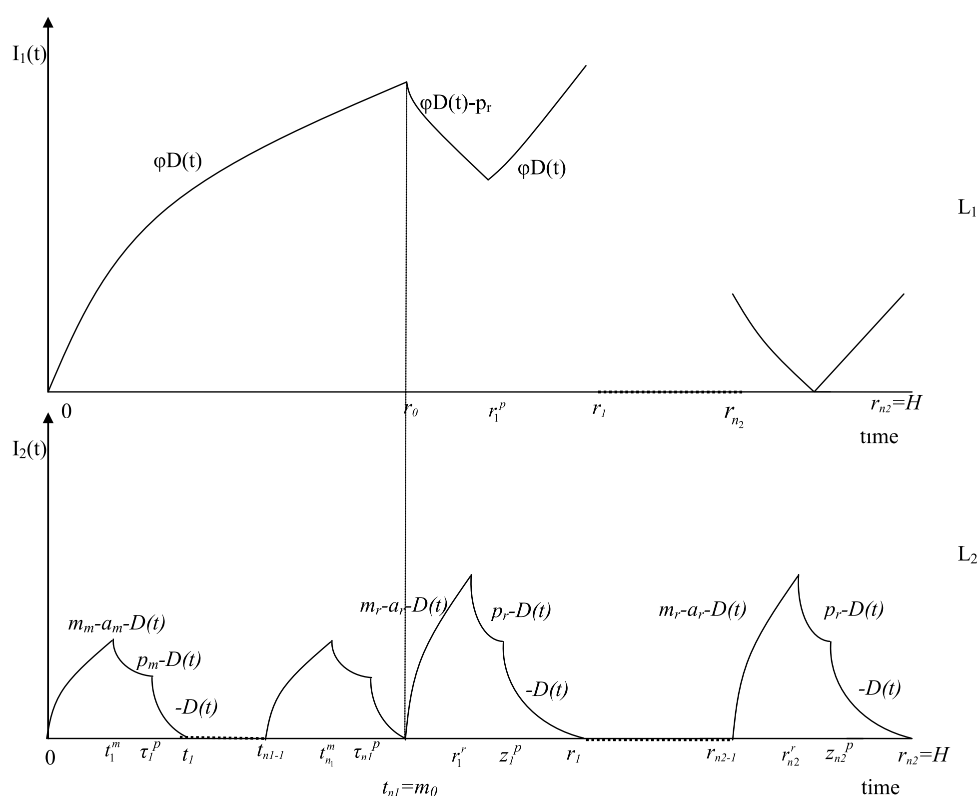

Fig. 2. Graphical representation of inventory level.

an EPQ model. There, it was assumed that $m_m - a_m$ is proportional to m_m, an assumption, which is relaxed in the present work.

Direct computations show that the amount of inventory $\int_{t_{i-1}}^{t_i^m} I_2(t)dt$ on the interval (t_{i-1}, t_i^m) is given by:

$$A_1 := \int_{t_{i-1}}^{t_i^m} (t_i^m - t)\{m_m - a_m - D(t)\}dt, \tag{4}$$

the amount of inventory $\int_{t_i^m}^{\tau_i^p} I_2(t)dt$ on the interval (t_i^m, τ_i^p) is given by:

$$A_2 := \int_{t_i^m}^{\tau_i^p} (t - t_i^m)\{D(t) - p_m\}dt + (\tau_i^p - t_i^m)\int_{\tau_i^p}^{t_i} D(t)dt, \tag{5}$$

and the amount of inventory $\int_{\tau_i^p}^{t_i} I_2(t)dt$ on the interval (τ_i^p, t_i) is given by:

$$A_3 := \int_{\tau_i^p}^{t_i} (t - \tau_i^p)D(t)dt. \tag{6}$$

Modelling considerations require, by continuity of the function I_2, that

$$(m_m - a_m)(t_i^m - t_{i-1}) + p_m(\tau_i^p - t_i^m) = \int_{t_{i-1}}^{t_i} D(t)dt, \tag{7}$$

and

$$m_m(t_i^m - t_{i-1}) = \int_{t_{i-1}}^{t_i} D(t)dt. \tag{8}$$

Using (7) and (8), it follows, after lengthy direct calculations, that the total inventory

$$\int_{t_{i-1}}^{t_i} I_2(t)dt = A_1 + A_2 + A_3$$

is equal to

$$H_2^m(t_{i-1}, t_i) := \int_{t_{i-1}}^{t_i} (t - t_{i-1})D(t)dt - \frac{1}{2\eta_m}\left\{\int_{t_{i-1}}^{t_i} D(t)dt\right\}^2, \tag{9}$$

where

$$\eta_m = \frac{p_m m_m^2}{a_m^2 + p_m(a_m + m_m)}. \tag{10}$$

The quadratic form (9) is also found in Omar and Smith (2002) in the context of optimal batching problems. A similar form can be obtained if one assumes that repair occurs within a regular production run, that is before τ_i as in Benkherouf and Omar (2017).

For the repair phase, the evolution of the inventory on the interval $[r_0, r_{n_2}]$ is similar to phase I with obvious differences. For a typical cycle k starting at time r_{k-1} and finishing at time r_k, with $k = 1, \ldots, n_2$ we have:

$$\frac{dI_2(t)}{dt} = m_r - a_r - D(t), \quad r_{k-1} \le t < r_k^r, \tag{11}$$

$$\frac{dI_2(t)}{dt} = -D(t) + p_r, \quad r_k^r \le t < z_k^p, \tag{12}$$

$$\frac{dI_2(t)}{dt} = -D(t), \quad z_k^p \le t < r_k, \tag{13}$$

with boundary conditions $I_2(r_{k-1}) = I_2(r_k) = 0$, and the function I_2 is continuous on the interval $[r_{k-1}, r_k]$. Again, the amount of inventory

$$\int_{r_{k-1}}^{r_k} I_2(t)dt, \tag{14}$$

is given by

$$H_2^r(r_{k-1}, r_k) := \int_{r_{k-1}}^{r_k} (t - r_{k-1})D(t)dt - \frac{1}{2\eta_r}\left\{\int_{r_{k-1}}^{r_k} D(t)dt\right\}^2, \tag{15}$$

where

$$\eta_r = \frac{p_r m_r^2}{a_r^2 + p_r(a_r + m_r)}. \tag{16}$$

It follows from (9) and (15) that the amount of inventory in L_2 on $[0, r_0]$ and $[r_0, r_{n_2}]$ are respectively

$$\sum_{i=1}^{n_1} H_2^m(t_{i-1}, t_i), \tag{17}$$

and

$$\sum_{k=1}^{n_2} H_2^r(r_{k-1}, r_k). \tag{18}$$

Let

$$H_2(t_1, \ldots, t_{n_1}, r_1, \ldots, r_{n_2}) := \sum_{i=1}^{n_1} H_2^m(t_{i-1}, t_i) + \sum_{k=1}^{n_2} H_2^r(r_{k-1}, r_k). \tag{19}$$

Using a similar argument as above and noting that for $i = 1, \ldots, n_1$ once t_{i-1} and t_i are known, then so are t_i^m and τ_i^p as the following relations indicate

$$t_i^m = t_{i-1} + \frac{1}{m_m}\int_{t_{i-1}}^{t_i} D(t)dt, \tag{20}$$

$$\tau_i^p = t_{i-1} + \frac{a_m + p_m}{p_m m_m}\int_{t_{i-1}}^{t_i} D(t)dt. \tag{21}$$

Similarly, for $k = 1, \ldots, n_2$, we have

$$r_k^r = r_{k-1} + \frac{1}{m_r}\int_{r_{k-1}}^{r_k} D(t)dt, \tag{22}$$

$$z_k^p = r_{k-1} + \frac{a_r + p_r}{p_r m_r}\int_{r_{k-1}}^{r_k} D(t)dt. \tag{23}$$

4.2. Evolution of inventory in location L_1

Location L_1 is a location dedicated to keep returned items for repair at some later time. During the interval $[0, r_0)$ stock of returned items builds up at a rate $\varphi D(t)$. However, during the repair stage, inventory depletes on some sub-intervals $(r_{k-1}, r_k^r]$ and increases on $(r_k^r, r_k]$. We assume that:

A1. Items left in Location L_1 at the end of planning horizon are either disposed off or sold to a secondary market.

The evolution of the inventory in this case is modeled by the equations:

$$\frac{dI_1(t)}{dt} = \varphi D(t), \quad 0 \le t < r_0, \tag{24}$$

$$\frac{dI_1(t)}{dt} = \varphi D(t) - m_r, \quad r_{k-1} \le t < r_k^r, \tag{25}$$

$$\frac{dI_1(t)}{dt} = \varphi D(t), \quad r_k^r \le t < r_k. \tag{26}$$

with appropriate boundary conditions. Using the same argument as in Benkherouf et al. (2014) and omitting details, the amount of inventory on the interval $[0, r_0)$ can be shown to be equal to

$$H_1^m(0, r_0) := \varphi\int_0^{r_0} (r_0 - t)D(t)dt. \tag{27}$$

The amount of inventory on the interval $[r_0, r_{n_2}]$ is equal to

$$\sum_{k=1}^{n_2-1} H_1^r(r_{k-1}, r_k) + \widetilde{H}_1^r(r_0, r_{n_2}), \tag{28}$$

where

$$H_1^r(r_{k-1}, r_k) := \frac{1}{2m_r}\left\{\int_{r_{k-1}}^{r_k} D(t)dt\right\}^2 + r_{k-1}\int_{r_{k-1}}^{r_k} D(t)dt - \varphi\int_{r_{k-1}}^{r_k} tD(t)dt, \tag{29}$$

and

$$\widetilde{H}_1^r(r_0, r_{n_2}) := (r_{n_2-1} - r_0)\Lambda - r_{n_2-1}(1-\varphi)\int_{r_{n_0}}^{r_{n_2-1}} D(t)dt, \tag{30}$$

with

$$\Lambda = \varphi\int_0^{r_0} D(t)dt. \tag{31}$$

The quantity Λ represents the maximum stock level in Location L_1. Set

$$H_1(t_1,\dots,t_{n_1},r_1,\dots,r_{n_2}) := H_1^m(0,r_0) + \sum_{k=1}^{n_2-1} H_1^r(r_{k-1},r_k) + \widetilde{H}_1^r(r_0,r_{n_2}). \tag{32}$$

Note that if $\varphi\int_0^H D(t)dt < \int_{r_0}^H D(t)dt$, then the amount consumed during the period $[r_0, H)$ is bigger than the amount collected during the whole planning horizon. This leads to inconsistency in the model. The following assumption is put forward to avoid that this happens.

A2. For a given planning horizon H, r_0 satisfies, $\varphi\int_0^H D(t)dt > \int_{r_0}^H D(t)dt$.

Under Assumption A2, the pair $(r_{n_2-1}, z_{n_2}^p)$ is found from solving the system of equations

$$\begin{aligned}
\varphi\int_0^{z_{n_2}^p} D(t)dt - \int_{r_0}^H D(t)dt &= 0,\\
m_r(z_{n_2}^p - r_{n_2-1}) - \int_{r_{n_2-1}}^H D(t)dt &= 0.
\end{aligned}$$

It follows that under assumptions A1 and A2 expression (32) is valid.

4.3. Total cost formulation

Exploiting the results of subsections (4.1)–(4.2) we are now able to compute the total cost of the model. For the carbon emission cost, generated by each of the system's activities, we use the formulation defined by Liu et al. (2015). In order to derive the total cost during the whole planning horizon, the individual costs are firstly derived.The sum, of set up cost and the corresponding carbon tax which is generated by carbon emissions in setup process, is:

$$(c_m + c_m^e\alpha)n_1 + (c_p + c_p^e\alpha)n_1 + (c_r + c_r^e\alpha)n_2 + (c_p + c_p^e\alpha)n_2.$$

By writing

$$K_m := c_m + c_m^e\alpha + c_p + c_p^e\alpha,\quad K_r := c_r + c_r^e\alpha + c_p + c_p^e\alpha,$$

the total set up cost is:

$$K_m n_1 + K_r n_2.$$

The collection cost along with the corresponding carbon tax is:

$$C_c := (u_f + u_f^e\alpha)\int_0^H \varphi D(t)dt,$$

the production cost which is made up of the manufacturing cost (cost of raw materials plus the cost to convert the materials into products) and the corresponding carbon tax:

$$\begin{aligned}
C_m(r_0) :&= (u_m + u_m^e\alpha)\sum_{i=1}^{n_1}\int_{t_{i-1}}^{\tau_i^p} m_m dt\\
&= (u_m + u_m^e\alpha)\int_0^{r_0} D(u)du,
\end{aligned} \tag{33}$$

the sum of remanufacturing cost and the corresponding carbon tax:

$$\begin{aligned}
C_r(r_0) :&= (u_r + u_r^e\alpha)\sum_{k=1}^{n_2}\int_{r_{k-1}}^{\tau_k^r} m_r dt\\
&= (u_r + u_r^e\alpha)\int_{r_0}^H D(u)du,
\end{aligned} \tag{34}$$

the sum of repair cost and the corresponding carbon tax is:

$$\begin{aligned}
C_p(r_0) &:= (u_p + u_p^e\alpha)\sum_{i=1}^{n_1}\int_{\tau_i^m}^{\tau_i^p} p_m dt + (u_p + u_p^e\alpha)\sum_{k=1}^{n_2}\int_{\tau_k^r}^{z_k^p} p_r dt\\
&= (u_p + u_p^e\alpha)\frac{a_m}{m_m}\int_0^{r_0} D(t)dt + (u_p + u_p^e\alpha)\frac{a_r}{m_r}\int_{r_0}^H D(t)dt,
\end{aligned} \tag{35}$$

and the sum of the holding cost and the corresponding carbon tax which is:

$$\begin{aligned}
H(t_1,\dots,t_{n_1},r_1,\dots,r_{n_2}) :=& (h_1 + h_1^e\alpha)[H_1(t_1,\dots,t_{n_1},r_1,\dots,r_{n_2})]\\
&+ (h_2 + h_2^e\alpha)[H_2(t_1,\dots,t_{n_1},r_1,\dots,r_{n_2})].
\end{aligned} \tag{36}$$

So the overall cost of the inventory system over $[0, H]$ is:

$$\begin{aligned}
TC(t_1,\dots,t_{n_1},r_1,\dots,r_{n_2}) :=& K_m n_1 + K_r n_2\\
&+ C_c + C_m(r_0) + C_r(r_0) + C_p(r_0) + H(t_1,\dots,t_{n_1},r_1,\dots,r_{n_2}).
\end{aligned} \tag{37}$$

5. Problem optimization

This section deals with the determination of the optimal manufacturing-remanufacturing-repairing schedule. That is, the determination of the values of $t_1,\dots,t_{n_1},r_1,\dots,r_{n_2}$ along with integer values of n_1 and n_2 are sought to minimize the total costs defined in (37) subject to the constraints $t_1 \le \cdots \le t_{n_1} \le r_1 \le \cdots \le r_{n_2}$, where $r_0 = t_{n_1}$, and $r_{n_2} = H$.

We shall follow a similar strategy adopted in Benkherouf et al. (2014), in addition to using some recent results on finite horizon models given in Benkherouf and Gilding (2020). The key idea is to consider the time r_0 (decision variable) as a parameter and to note that for fixed r_0, expressions (33)–(35) are fixed. The results below show that the quest for the optimum inventory policy can be undertaken on two independent optimization problems, which are structurally identical to the optimization problems treated in Section 5 of Benkherouf et al. (2014). Therefore, a number of results will be stated without Proof, interested readers may consult Benkherouf et al. (2014).

Theorem 5.1. *For fixed m_0 and under assumptions **A1** and **A2** the optimal inventory policy reduces to solving mixed integer nonlinear programs **P1** and **P2** with*

P1:

$$\min z_1 = K_m n_1 + \sum_{i=1}^{n_1} R_1(t_{i-1}, t_i) \tag{38}$$

subject to

$$0 = t_0 \le \cdots t_{n_1} = r_0, \tag{39}$$

$$n_1 \ge 1, \tag{40}$$

where, by (9) and (36),

$$R_1(t_{i-1}, t_i) = c_1 \left[\int_{t_{i-1}}^{t_i} (t - t_{i-1}) D(t) dt - \frac{1}{2\eta_m} \left\{ \int_{t_{i-1}}^{t_i} D(t) dt \right\}^2 \right], \tag{41}$$

with $c_1 = h_1 + h_1^e \alpha$ and

P2:

$$\min z_2 = K_r(n_2 - 1) + \sum_{k=1}^{n_2-1} R_2(r_{k-1}, r_k) \tag{42}$$

subject to

$$r_0 \le r_1 \le \cdots \le r_{n_2-1}, \tag{43}$$

$$n_2 \ge 2, \tag{44}$$

where, by (15) and (36),

$$R_2(r_{k-1}, r_k) = (c_1 - c_2) \left[\int_{r_{k-1}}^{r_k} (t - r_{k-1}) D(t) dt - \frac{1}{2\eta_r} \left\{ \int_{r_{k-1}}^{r_k} D(t) dt \right\}^2 \right], \tag{45}$$

with $c_2 = h_2 + h_2^e \alpha$.

If $c_1 - c_2 > 0$ then **P1** and **P2** are similar. The general problem was primarily dealt with in Al-Khamis et al. (2014). The next subsections will be concerned with their solutions. Furthermore, if r_0 is known, then under assumption **A2** the value of r_{n_2-1} is known, which justifies the fact that the summation in the objective function in problem **P2** runs up to $n_2 - 1$.

5.1. Convexity results and existence of the optimal solution

Note that $c_1 - c_2$ can be dropped from **P1** and **P2** if $c_1 - c_2 > 0$ leading to the examination of the following generic problem which puts **P1** and **P2** into one parcel. The case $c_1 - c_2 < 0$ will be treated in the next subsection.

For a given $h > 0$, $\eta > D(t)$, and $K > 0$, consider.

P:

$$\min \; z = Kn + \sum_{j=1}^{n} R(s_{j-1}, s_j) \tag{46}$$

subject to

$$0 \le s_1 \le \cdots \le s_n = h, \tag{47}$$

$$n \ge 1, \tag{48}$$

where

$$R(x, y) = \int_x^y (t - x) D(t) dt - \frac{1}{2\eta} \left\{ \int_x^y D(t) dt \right\}^2. \tag{49}$$

Recall that a differentiable function R is strictly sub-modular if either (i) $\partial_x R(x, y) < 0$ is strictly decreasing in y for $x < y$ or (ii) $\partial_y R(x, y) < 0$ is strictly decreasing in x for $x < y$ (see Topkis (1998)).

Lemma 5.2. *The function R defined in (49) is strictly sub-modular.*

Proof. Note that

$$\partial_y R(x, y) = \left\{ \left(y - x \right) - \frac{1}{\eta} \int_x^y D(t) dt \right\} D(y),$$

and

$$\partial_x \partial_y R(x, y) = -\left\{ 1 - \frac{D(x)}{\eta} \right\} D(y).$$

The lemma is then immediate since $D(x) < \eta$. For fixed n, consider the problem of finding the minimizers of

$$S(s_1, \ldots, s_n) = \sum_{j=1}^{n} R(s_{j-1}, s_j). \tag{50}$$

subject to constraint (47).

Lemma 5.3. *There exists a solution which minimizes $S(s_1, \ldots, s_n)$ subject to (47). Moreover, this solution is a stationary point of S.*

Proof. The existence stems from the fact that the function S is continuous on a compact set. A minimizer necessarily satisfies $0 < s_1 < \cdots < s_n = h$. To finish the proof, let us examine the problem of finding γ which minimizes the function σ defined by:

$$\sigma(\gamma) = R(x, \gamma) + R(\gamma, y),$$

with $0 \le x \le \gamma \le y \le h$. It follows that $\sigma'(\gamma) = (\partial_x) R(x, \gamma) + (\partial_y) R(\gamma, y)$. But $\sigma'(x) = (\partial_x) R(x, x) + (\partial_y) R(x, y) < (\partial_x) R(x, x) + (\partial_y) R(x, x)$. But the last term is equal to zero. Therefore, $\sigma'(x) < 0$. The same argument shows that $\sigma'(y) > 0$. Hence, the minimum of $\sigma(\gamma)$ occurs on the interior of interval (x, y). This leads to the required result.Define $S_n^*(h)$ be the minimum value of the objective function defined in (50). The next theorem is found in Benkherouf and Gilding (2020).

Theorem 5.4. *The function $S_n^*(h)$ satisfies*

$$S_{n+2}^*(h) - S_{n+1}^*(h) \ge S_{n+1}^*(h) - S_n^*(h).$$

Remark 1.

(i) Theorem 5.4 shows that the function S^* is convex in n. This property makes the determination of the optimal value of n relatively easy: for more on the implication of this property see Benkherouf et al. (2014) and Benkherouf and Gilding (2009).

(ii) Note that Theorem 5.4 is valid with only generic conditions. The theorem as presented is the strongest existing theorem in the literature for such models. Related results were shown under the existence of a unique of the minimizer of $S(s_1, \ldots, s_n)$ subject to (47).

(iii) Lemma 5.3 and Theorem 5.4 can now be used to find the optimal solution for **P1** and **P2**. Indeed, a solution of **P1** can be determined using any of the shelves optimization softwares. The optimal value of n is then found using the convexity property along the same lines as in Benkherouf et al. (2014) and Benkherouf and Gilding (2009).

We shall next examine the case where the solution of the optimization is unique.

5.2. Uniqueness and strict convexity

In addition to assumptions **A1** and **A2** we assume that.
A3. The function D is log-concave on $[0, H]$.
A4. The function

$$\nu(x) := \frac{D'(x)}{1 - D(x)\eta^{-1}},$$

is non-decreasing in x on $[0, H]$.

The following result is found in Benkherouf et al. (2014).

Lemma 5.5. *Under assumptions **A1**–**A4**, there exists a unique solution which minimizes $S(s_1, \ldots, s_n)$ subject to (47). Moreover, this solution is a stationary point of S.*

Define $S_n^*(h)$ be the minimum value of the objective function defined in (50) under the hypothesis of Lemma 5.5.

Theorem 5.6. *The function $S_n^*(h)$ satisfies*

$$S_{n+2}^*(h) - S_{n+1}^*(h) > S_{n+1}^*(h) - S_n^*(h).$$

It is worth noting that Theorem 5.6 deals with strict convexity whereas Theorem 5.4 relates to weak convexity.

The next result, which is stated without Proof and which relates the length of successive intervals in a cycle, may also be found useful in a numerical search for the optimal solution or managerial interpretation of the optimal inventory policy.

Lemma 5.7. *Let $s_1, \ldots, s_n$ is the optimal solution of $S(s_1, \ldots, s_n)$ subject to (47) under the hypothesis of Lemma 5.5. For $i = 1, \ldots, n-1$,*

(i) *if D is strictly increasing, then $s_{i+1} - s_i < s_i - s_{i-1}$.*
(ii) *if D is strictly decreasing, then $s_{i+1} - s_i > s_i - s_{i-1}$.*
(iii) *if D is constant, then $s_{i+1} - s_i = s_i - s_{i-1}$.*

The proceeding results are now ready to be used to solve problems **P1** and **P2** when $c_1 - c_2 > 0$, as Lemma 5.5 and Theorem 5.4 are only applicable with the generic hypothesis of the model. If the inventory analyst is interested uniqueness, then assumptions **A3** and **A4** are required in which case Lemmas 5.5, 5.7, and Theorem 5.6 apply.

5.3. The case $c_1 - c_2 < 0$

Recall that this case applies only to Problem **P2** and consider a companion the mixed integer nonlinear program **P** discussed in the previous subsection where the function R in the function is replaced with the function $\widetilde{R}$, to get the problem.

$\widetilde{\mathbf{P}}$:

$$\min \; z = Kn + \sum_{j=1}^{n} \widetilde{R}(s_{j-1}, s_j) \tag{51}$$

subject to

$$0 \le s_1 \le \cdots \le s_n = h, \tag{52}$$

$$n \ge 1, \tag{53}$$

where

$$\widetilde{R}(x, y) = -R(x, y),$$

with R given in (49), $h > 0$, $\eta > D(t) > 0$, and $K > 0$.

For fixed n, consider the problem of minimizing

$$\widetilde{S}_n(s_1, \ldots, s_n) = \sum_{j=1}^{n} \widetilde{R}(s_{j-1}, s_j),$$

under constraint (52).

The function $\widetilde{S}_n$ admits a minimizer on the subset of $\mathbb{R}^n$ defined by the constraint. This follows from the fact that $\widetilde{S}_n$ continuous on a compact set. Moreover, this minimizer is not necessarily a critical point and the second part of Lemma 5.5 is no longer valid. Moreover, the convexity property is replaced by a concavity property as the following result shows.

Theorem 5.8. *Let $\widetilde{S}^*{}_n(h)$ be the minimum of the objective value of $\widetilde{S}_n(s_1, \ldots, s_n)$ under constraint (52), then*

$$\widetilde{S}^*{}_{n+2}(h) - \widetilde{S}^*{}_{n+1}(h) \le \widetilde{S}^*{}_{n+1}(h) - \widetilde{S}^*{}_n(h).$$

The Proof of Theorem 5.8 is omitted and is similar in spirit to that of Theorem 5.4.

Now if assumptions **A1** and **A2** are valid, then the argument used in Benkherouf et al. (2014) applies. That is, the maximum of $\widetilde{S}_n(s_1, \ldots, s_n)$ under constraint (52) is unique and occurs at the critical point and the minimum is $n = 1$ and $s_1 = \cdots = s_n = h$.

5.4. Special cases

If we set specific values in particular parameters of our model, then the models summarized in Table 2 are recovered, in terms of the total cost formulation .

6. Computational results

The proposed model, studied so far, could be applicable to any system that meets the characteristics outlined in Section 4 of this paper. Specifically, the model could reflect forms of industry as laser printer cartridges, mobile phones and car spares. The aim of this section is to gain, through numerical examples, managerial insights that mainly concerns the sustainability related parameters of the system. The basic numerical example mainly figures the traditional lead–acid battery industry (based on the works of Taleizadeh et al. (2019) and Jauhari et al. (2020). This type of batteries is commonly used in automobile industry and consists of two primary elements, the pure lead, acting as a negative electrode, and the lead dioxide, acting as a positive electrode. Both manufacturing and remanufacturing processes are commonly used to create new lead acid batteries. Casting, pasting, curing, assembling, drying, and stamping are all used in the production of lead acid batteries. Because they produce carbon emissions, some of these processes are harmful. Remanufactured batteries are sold in the primary market and are of the same quality as manufactured ones.

6.1. Numerical example

In this section we shall present some numerical examples to demonstrate the model applicability and to examine the effect of carbon tax constraints on the inventory policy. Exponential demand rate and linear decreasing rate functions were chosen because they are the most widely used types of demand rate in the related literature. The parameters for the used demand rates $D(t)$ were selected to satisfy the assumptions **A1**–**A4** and in addition, to make the relations $\eta_m > D(t)$ and $\eta_r > D(t)$ valid. Furthermore, the carbon emissions per product unit during

Table 2
Models arising as special cases from the present model.

Parameters	Models
$\alpha = 0,\ \varphi = 0,\ h_1 = 0,\ D(t) = D$	Jamal et al. (2004)
$\alpha = 0,\ p_m = 0,\ p_r = 0$	Benkherouf et al. (2014)
$\alpha = 0,\ \varphi = 0,\ h_1 = 0$	Benkherouf and Omar (2017)
$\alpha = 0,\ \varphi = 0,\ h_1 = 0,\ a_m = 0,\ a_r = 0,$ $D(t) =$ linearly increasing	Hong et al. (1990)
$\alpha = 0,\ m_r \to \infty,\ \varphi = 0,\ h_1 = 0,\ a_m = 0,\ a_r = 0,$ $D(t) =$ linearly increasing	Resh et al. (1976) and Donaldson (1977)
$\alpha = 0,\ m_r \to \infty,\ \varphi = 0,\ h_1 = 0,\ a_m = 0,\ a_r = 0,$ $D(t) =$ linear	Hariga (1993)

Table 3
The optimal policy for exponential (basic scenario) and linear demand rates.

Demand	t_0	t_1	t_2	t_3	r_0	r_1
$10\exp(0.2t)$	0	1.47	2.90	4.33	4.33	5
$30-2t$	0	3.82	–––	–––	3.82	5

the remanufacturing process are chosen to be 50% lower than that of the manufacturing process assuming that the remanufacturing process has less energy usage. Although the per unit remanufacturing cost is lower than the manufacturing cost, we assume that the whole remanufacturing process is the costliest operation, since the cost covers also the collection activities. This is particularly true when the return flow is ill-defined and the recovery of end-of-life products is costly, or if recovered products need pre-treatment to standardize the quality of the material before remanufacturing begins (García-Alvarado et al., 2017). Based on the above described rationale, which is similar to García-Alvarado et al. (2017) and Anthony and Cheung (2017), the data set used as basic scenario, along with the proper adaptations, is: $u_f = 7$, $u_f^e = 1$, $\varphi = 0.2$, $\alpha = 20$, $D(t) = 10e^{0.2t}$, $H = 5$, $m_m = 50$, $m_r = 45$, $a_m = 12$, $a_r = 8$, $p_m = 8$, $p_r = 6$, $c_r = 35$, $c_m = 15$, $c_p = 2.5$, $u_r = 8$, $u_m = 10$, $u_p = 9$, $c_r^e = 1$, $c_m^e = 2$, $c_p^e = 1$, $u_r^e = 1$, $u_m^e = 2$, $u_p^e = 1.5$, $h_1 = 5$, $h_2 = 1.5$, $h_1^e = 1.5$ and $h_2^e = 0.5$.

The procedure to solve the problem is: A search for r_0 is conducted using 0.01 step. The search range is defined by relationship $\varphi \int_0^H D(t)dt > \int_{r_0}^H D(t)dt$. For each r_0, the optimum solution and cost for the two sub-problems (**P1** and **P2**, say $z_1(r_0)$ and $z_2(r_0)$) are obtained along with the manufacturing/repairing and remanufacturing/repairing lots. Finally, the optimal r_0 is: $r_0^* = argmin_{r_0}(z_1(r_0) + z_2(r_0))$, which leads to the overall optimal policy.

Using the previous procedure, we solve our basic numerical example along with the one with linear demand rate. The optimal policy is shown in Table 3. According to Table 3 the optimal replenishment policy, for the exponential demand rate, requires three (3) manufacturing and one (1) remanufacturing runs while for the linear demand rate, one (1) manufacturing and one (1) remanufacturing runs, respectively. In addition the results of Lemma 5.7, which relate the length of successive intervals in a cycle, are satisfied. The results of Table 3 indicate the high impact of demand rate on optimal policy and cost. Also, the remanufacturing process starts later when the demand rate is exponential perhaps in this way the system has the ability to accumulate greater quantity of returned products. Table 4 presents the values for the optimal quantities Q_i^j and the total carbon emissions, where Q_i^j is the quantity produced in cycle i by the process j, $j = m$ for manufacturing, $j = p_m$ for repair during manufacturing process, $j = r$ for remanufacturing and $j = p_r$ for repair during the remanufacturing process.

6.2. Managerial insights based on sensitivity analysis

Using the data of the basic scenario (under exponential demand rate) post optimality analysis is conducted. To this end, the impact of system main parameters changes are considered on total cost. Fig. 3 represents the changes of optimal cost with respect to α for different values of return rates (φ). When α increases (for the same value of φ) a high increase in cost is observed. In spite of the fact that the changes in φ alter the production schedule (since φ impacts the determination of m_0) the impact of changes in system cost's operation is minor. On the contrary, the increase in α does not affect the optimal policy, however, the increase in optimal cost is considerable high (more than 160% increase). This is an evidence that investment in green technology could not only be an indication of social responsibility but also profitable. In addition, these results could be used as a guide helping the decision maker to determine both the amount of investment and depreciation period in green technology under different scenarios for carbon tax level.

From Fig. 4 it is observed that the model is fairly robust to changes in all the processes that are used for the demand satisfaction (as expressed through, m_m, m_r, p_m, p_r parameters) and this behavior is consistent for all values of α. However, the exploitation of eco-conscious manufacturing processes seems to be profitable (i.e. as each of m_m and m_r increases, the cost decreases). Inevitably, the high impact of α on optimal cost should be again highlighted.

As expected, Fig. 5 indicates that as the number of defective items increases (either during the manufacturing or during the remanufacturing process) the optimal cost increases and this is higher for higher values of α. Again, the impact of α on cost is significant. These findings hint that investment on quality control processes and on production of high-quality items is always a priority that could lead to cost reduction. Notice that the repair processes seem to be more costly in tax increase in relation to other processes.

Fig. 6 shows that when the carbon emissions from the system activities increase, then the optimal cost increases, which is reasonable. Also, it seems that the increase in optimal cost is more sensitive to the changes of the carbon emitted from the manufacturing process in relation to the other processes (i.e. remanufacturing and repairing). These results build on some existing evidence of related literature (Sundin and Lee (2012)).

Closing this section, we can conclude that the proposed model provides guidelines for decision makers to build an appropriate strategy for their production-inventory systems. The production-inventory schedule for each process (manufacturing, remanufacturing and repairing) can be determined by taking into account the carbon emissions emitted from

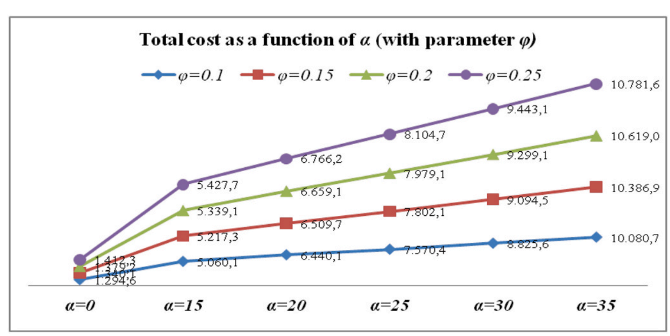

Fig. 3. Sensitivity analysis for collection rate.

Table 4
The optimal quantities produced by the three processes and the total carbon emissions for exponential (basic scenario) and linear demand rates.

Demand	Q_1^m	$Q_1^{p_m}$	Q_2^m	$Q_2^{p_m}$	Q_3^m	$Q_3^{p_m}$	Q_1^r	$Q_1^{p_r}$	Total carbon emissions
$10\exp(0.2t)$	17.47	4.17	21.40	5.28	29.80	7.10	17.24	3.03	263.995
$30-2t$	100	23.92	––	––	––	––	25	4.41	370.943

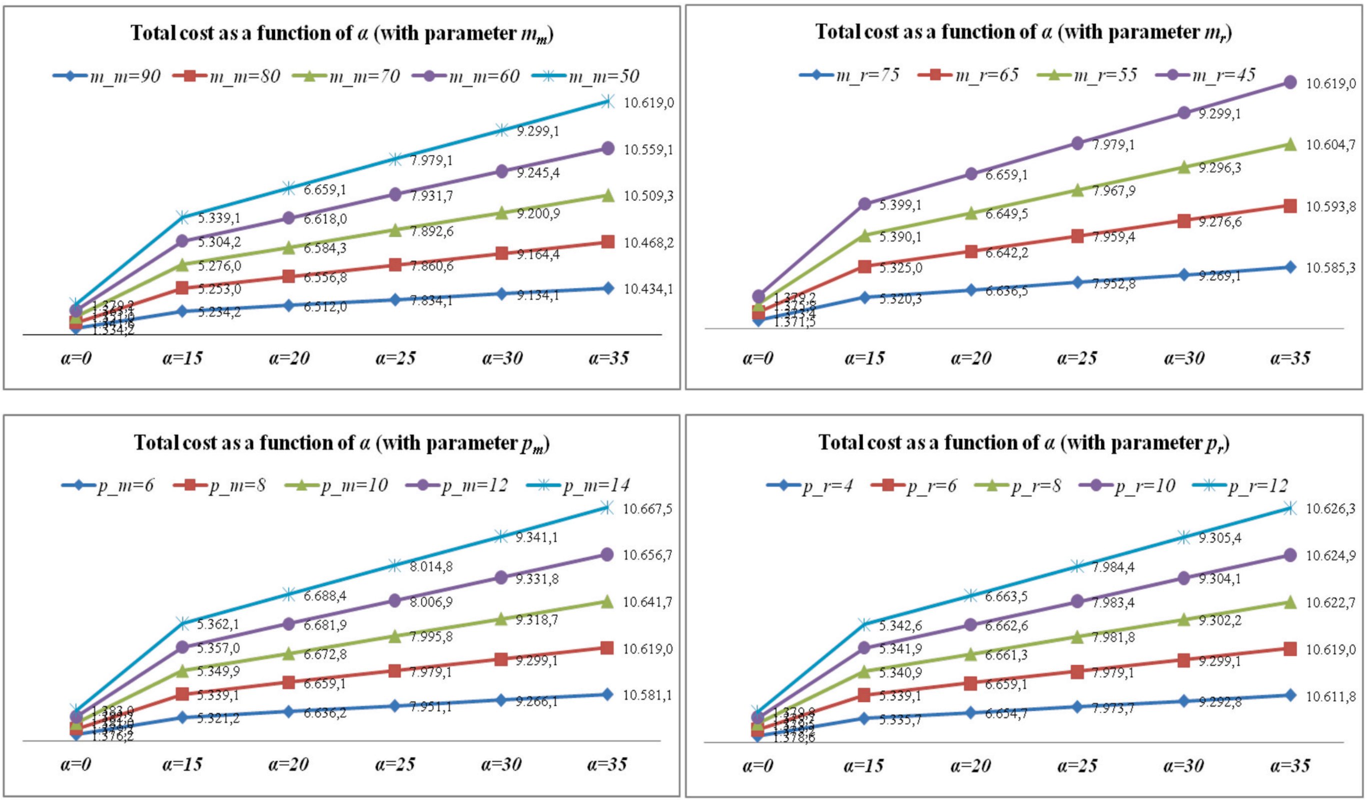

Fig. 4. Sensitivity analysis for manufacturing, remanufacturing and repairing rates.

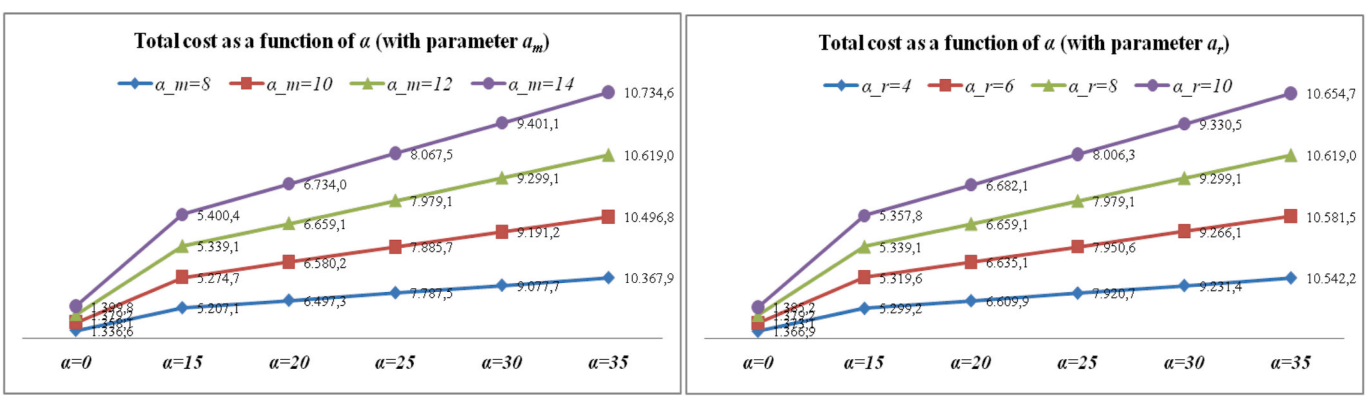

Fig. 5. Sensitivity analysis in manufacturing of defective items (during manufacturing and remanufacturing processes).

them. The decision makers can regulate emissions from manufacturing, remanufacturing and repairing processes in order to conform with the carbon tax regulations, adopting the most effective form of green technology.

7. Conclusions and prospects

The present paper aims to make a contribution to the field of green supply chain through efficient inventory management. Here, a continuous review production-inventory model is proposed with the options of remanufacturing and repairing activities under carbon tax regulation. In this context, the decision maker is required to determine the production scheduling over a finite planning horizon for a time varying demand. Produced items are allowed to be returned (this might be due to defects or other causes) and stored in a recoverable warehouse (location L_1). These items are remanufactured up to a time point of the planning horizon (a decision variable). During this period the demand is satisfied only by new manufactured lots of unequal size whereas in the rest of the planning horizon, demand is satisfied only by remanufactured lots of unequal size. Defective items can be produced during the manufacturing and remanufacturing phase. These items are repaired and transformed into perfect quality items, after the end of each manufacturing and remanufacturing phases. Carbon emissions generated by every process (manufacturing, remanufacturing, repairing, collection of used products) are charged the corresponding carbon tax cost in line with standard practices of green supply chain. The objective of the proposed model is to determine the optimal manufacturing, remanufacturing and repairing schedule that minimizes the total system operation cost. Therefore, the optimal policy involves computing the switching time point from manufacturing to remanufacturing and the manufacturing, remanufacturing and repairing lots.

For this purpose: (1) The problem of determining an optimal

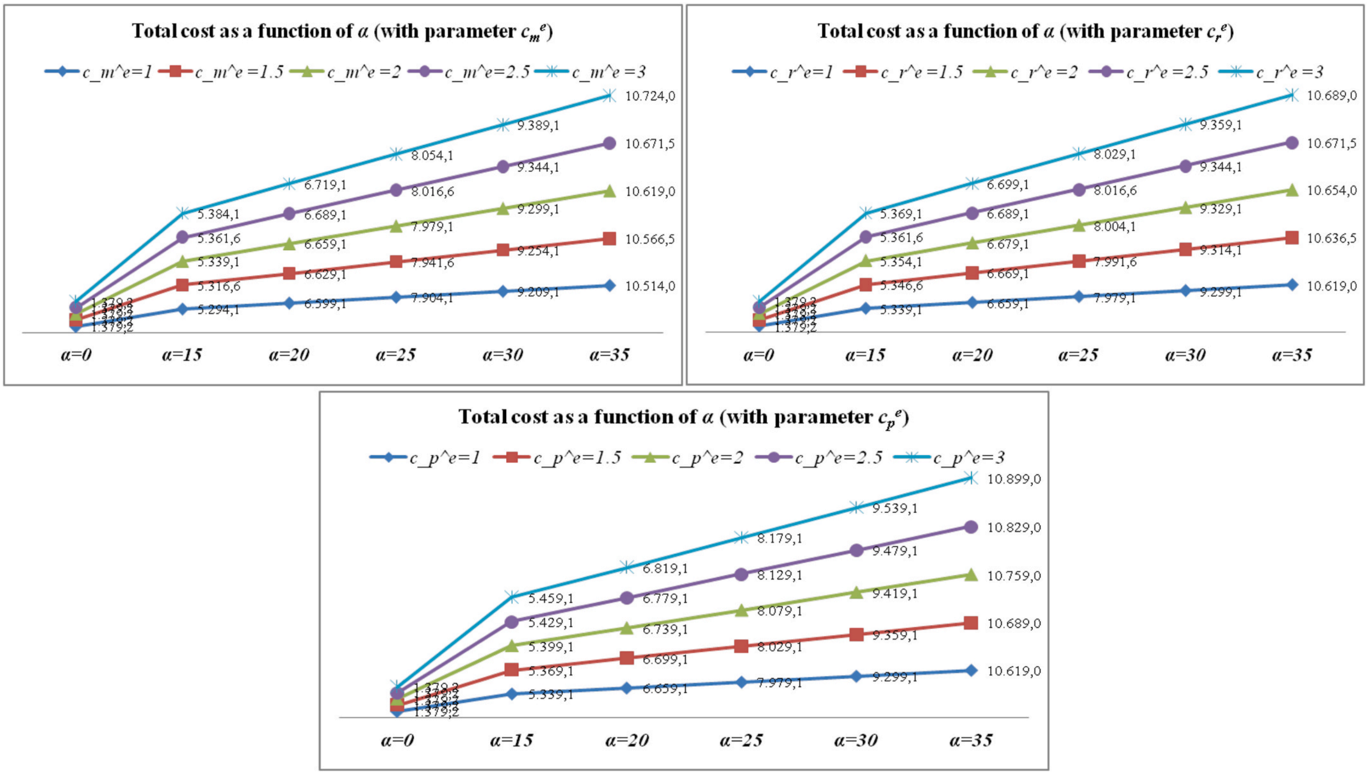

Fig. 6. Sensitivity analysis for carbon emitted from manufacturing, remanufacturing and repair processes.

production plan was formulated as a mixed integer nonlinear programming problem (MINLP). (2) An appropriate pre-processing of the MINLP problem led to the decomposition of the joint manufacturing/remanufacturing MINLP into two separate sub-problems: a pure manufacturing optimization problem and a remanufacturing one. (3) The existence of the optimal solutions of these two sub-problems was shown, while the uniqueness of the solutions was proved under specific conditions. (4) As a by-product previous known results in the literature were generalized and new ones were obtained. (5) An easy to use procedure, which relies on a finite simple search algorithm, was suggested for determining the overall optimal problem solution.

7.1. Implications for theory and practice

From a theoretical standpoint, the MINLP problem examined in the paper provided a challenge for devising a simple scheme to solve it. Numerical based approaches could have been used with no guarantee of achieving global optimal solution. The solution approach proposed in this paper ensures that an exact optimal solution scheme for the problem could be derived. This may pave the way for the resolution of more complex models that share similar structural properties. Also, it is worth noting that the optimization problems from different Operations Research areas share similar structural properties. For example, a problem related to the timings of manufacturing (remanufacturing) of this paper is hidden in Denardo et al. (1982). So, it is our hope that the present results find applications in other optimization areas.

From a practical point of view, this study could serve as a decision making tool for the manufacturing/remanufacturing firms of say, lead-acid batteries, laser printer cartridges, mobile phones and car spares providing an optimal strategic decision on the storage and the production planning of new, repaired and remanufactured products. Thus, hopefully, provide a step towards a better understanding of how inventory management responds to environmental and sustainability requirements. Also, note that the numerical results highlight that an increase in the tax charged per unit of CO_2 emitted does not affect the structure of the optimal policy (manufacturing and remanufacturing schedule), however the increase in optimal cost is considerably high (more than 160% increase). This is an indication of the importance of investments in green technology towards reducing carbon emissions. Moreover, it seems that the increase in optimal cost is more sensitive to the changes of the carbon emitted from the manufacturing process in relation to the other processes i.e. remanufacturing and repairing.

Finally, it is our hope that the theoretical results (worth noting that they are not trivial) and the managerial insights obtained can contribute to a better understanding of the impact of both carbon emission and combination of manufacturing, repairing and remanufacturing activities on the optimal policy for finite horizon inventory problems.

7.2. Limitations and further research

It is obvious that our research work is not exonerated from limitations. It is assumed that all of the used items that are collected from the primary market are remanufactured and considered as new ones. However, the quality of used items is not the same. So an interesting future research direction would be to include different quality conditions of the used items. Another possibility is to include pricing decisions and/or other type of carbon emissions reduction policies. In addition, the present model may be expanded by including demand and production disruptions (see Xu et al. (2016) and Konstantaras et al. (2019)). Finally, the results obtained in this paper may be used to create more efficient algorithms to solve structurally similar problems. They may also be used to devise heuristic algorithms to obtain quicker computational solutions for more complex problems.

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