Capacities of Gaussian Quantum Channels With Passive Environment Assistance

Samad Khabbazi Oskouei, Stefano Mancini, and Andreas Winter

***Abstract*— Passive environment-assisted communication takes place via a quantum channel modeled as a unitary interaction between the information carrying system and an environment, where the latter is controlled by a passive helper, who can set its initial state such as to assist sender and receiver, but not help actively by adjusting her behaviour depending on the message. Here we investigate the information transmission capabilities in this framework by considering Gaussian unitaries acting on Bosonic systems. We consider both quantum communication and classical communication with helper, as well as classical communication with free classical coordination between sender and helper (conferencing encoders). Concerning quantum communication, we prove general coding theorems with and without energy constraints, yielding multi-letter (regularized) expressions. In the search for cases where the capacity formula is computable, we look for Gaussian unitaries that are universally degradable or anti-degradable. However, we show that no Gaussian unitary yields either a degradable or anti-degradable channel for all environment states. On the other hand, restricting to Gaussian environment states, results in universally degradable unitaries, for which we thus can give single-letter quantum capacity formulas. Concerning classical communication, we prove a general coding theorem for the classical capacity under an energy constraint, given by a multi-letter expression. Furthermore, we derive an uncertainty-type relation between the classical capacities of the sender and the helper, helped respectively by the other party, showing a lower bound on the sum of the two capacities. Then, this is used to lower bound the classical information transmission rate in the scenario of classical communication between sender and helper.**

***Index Terms*— Quantum Gaussian channels, quantum Gaussian capacity, super-activation.**

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I. Introduction

In QUANTUM mechanics, every noisy channel (completely positive and trace preserving – CPTP – linear map) is the marginal of a reversible (i.e. unitary) interaction with an environment initially in a pure state; this is the content of Stinespring's dilation theorem [30], and of the subsequent structure theorems of Choi [6], Jamiołkowski [15] and Kraus [18]. This feature, which distinguishes quantum communication fundamentally from its classical counterpart, is at the core of the possibility to perform unconditional secret key agreement over a channel, since the channel essentially uniquely determines the action on the environment. In this picture, noise in the channel is entirely due to loss of information into the environment, more precisely the build-up of correlations between the system and the environment. A series of prior work, starting with [9], [10] have asked how much one can counteract the noise if one had access to the environment output state and could feed classical information back into the channel output system [11], [20], [21], [29], [34].

Somewhat dually, two of the present authors have asked previously what benefit can be obtained by accessing the *initial* state of the environment [16], [17]. In contrast to the (active) interventions in the environment of the aforementioned works, we call this passive environment-assistance, since the role of the helper is restricted to choosing a suitable initial state. These previous results were obtained in the finite-dimensional setting. Here, we extend the model and results to infinite-dimensional systems, with special attention to Gaussian channels and their Gaussian unitary dilations. Additional motivations for the model of passive environment-assistance comes from the notion of *environment-parametrized quantum channels*, which are used to describe quantum memory cells [8].

The present paper is structured as follows: In Section II we define the system model and establish basic notation. In Section III we treat quantum communication capacities both without and with energy constraints; we show that two-mode Gaussian unitaries are never universally degradable or anti-degradable, but restricting to Gaussian helper there are families of either type, allowing us to explicitly calculate the passive environment-assisted quantum capacity under this restriction. In Section IV we analyze the classical capacity with a helper under energy constraints both for sender and helper; we show that the capacity of the sender assisted by the helper and of the helper assisted by the sender cannot both be small, and apply this insight to the case of conferencing encoder. In Section V we prove that the passive environment-assisted quantum and classical capacities with energy constraints are continuous

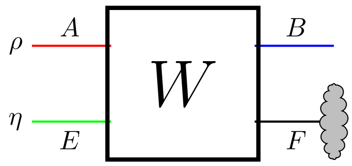

Fig. 1. Diagrammatic view of the parties involved in the communication. Sender (Alice) and receiver (Bob) control respectively the systems A and B. A third party controls the environment input system E to assist the communication between sender and receiver. The inaccessible output-environment system is labelled as F. All in all the quantum channel $\mathcal{N}^{AE\to B}$ is established.

in the unitary interaction, and indeed uniformly so with respect to the energy-constrained diamond norm. In Section VI we conclude. Two appendices provide additional proofs: In Appendix A we prove Theorem 7, stating that non-trivial two-mode Gaussian unitaries are neither universally degradable nor universally anti-degradable; Appendix B proves tighter lower bounds on the sum of classical capacities for two-mode Gaussian unitaries.

II. System Model and Notation

In the present paper, we consider a communication model between Alice and Bob that involves also a third party (helper) controlling the environment input system, whose aim is to enhance the communication between Alice and Bob (see Fig. 1). We assume that the helper sets the initial state of the environment to enhance the communication from Alice to Bob, and then has no role in the coding protocol, thus we refer to this model as *passive environment-assisted model*.

We will be interested in the realization of this model by infinite dimensional systems, particularly with Gaussian unitary transformations W and then to the channels that arise from them depending on the state η.

Let $\mathcal{L}(X)$ denote the space of linear operators on a (separable) Hilbert space X. We denote the identity operator in $\mathcal{L}(X)$ as $\mathbb{1}_X$ and the identity map (ideal channel) $\mathrm{id} : \mathcal{L}(X) \to \mathcal{L}(X)$ is denoted by id_X. For any linear operator $\Lambda : A \to B$ between Hilbert spaces we denote the trace norm

$$\|\Lambda\|_1 := \operatorname{Tr}\sqrt{\Lambda^\dagger\Lambda} = \operatorname{Tr}|\Lambda|, \tag{1}$$

and the operator norm

$$\|\Lambda\|_\infty := \sup\big\{\big|\Lambda|\psi\rangle\big| : |\psi\rangle \in A, \big||\psi\rangle\big| = 1\big\}, \tag{2}$$

where $|\cdot|$ denotes the Hilbert space norm. Let $\mathcal{T}(X) \subset \mathcal{L}(X)$ denote the set of trace class operators whose trace norm, defined above, is finite; likewise, $\mathcal{B}(X) \subset \mathcal{L}(X)$ is the set of bounded operators, whose operator norm is finite. Any positive semidefinite element $\rho \in \mathcal{T}(X)$ with $\operatorname{Tr}\rho = 1$ is called a density operator. Obviously the set $\mathcal{S}(X)$ of such operators is a proper subset of $\mathcal{T}(X)$. A quantum channel $\mathcal{N}$ from system A to a system B is a completely positive and trace preserving (CPTP) linear map from $\mathcal{T}(A)$ to $\mathcal{T}(B)$.

Coming back to Fig. 1, consider an isometry $W : A\otimes E \to B \otimes F$ which defines a channel $\mathcal{N} : \mathcal{L}(A \otimes E) \to \mathcal{L}(B)$, whose action on the input state σ on $A \otimes E$ is

$$\mathcal{N}^{AE\to B}(\sigma) = \operatorname{Tr}_F W\sigma W^\dagger. \tag{3}$$

Then an effective channel $\mathcal{N}_\eta : \mathcal{L}(A) \to \mathcal{L}(B)$ is established between Alice and Bob once the initial state η on E is set:

$$\mathcal{N}_\eta^{A\to B}(\rho) := \mathcal{N}^{AE\to B}(\rho\otimes\eta). \tag{4}$$

The complementary channel is

$$\widetilde{\mathcal{N}}_\eta^{A\to F}(\rho) := \operatorname{Tr}_B W(\rho\otimes\eta)W^\dagger, \tag{5}$$

while the adjoint channel $\mathcal{N}_\eta^{*\,B\to A}$ acts on the bounded operator $b \in \mathcal{B}(B)$ such that

$$\operatorname{Tr}\big[\mathcal{N}_\eta^{*\,B\to A}(b)\ \rho\big] = \operatorname{Tr}\big[b\mathcal{N}_\eta^{A\to B}(\rho)\big] \qquad b \in \mathcal{B}(A). \tag{6}$$

It can be written in the explicit form

$$\mathcal{N}_\eta^{*\,B\to A}(b) = \operatorname{Tr}_E W^\dagger(b\otimes \mathbb{1}_F)W(\mathbb{1}_A\otimes\eta), \tag{7}$$

using the isometry W and the state η.

The consideration of the model of Fig. 1 is motivated also by the fact that it can give insights for quantum multiple-access channels whose characterization is usually quite challenging. Indeed the channel (3) can equivalently be seen as quantum multi-access channels (two-sender-one-receiver) [36]. Additionally, the channel (4) can be intended as an environment-parametrized quantum channel and hence can be used to describe quantum memory cells [8] and to characterize quantum reading capacity [22].

Since we will deal with infinite dimensional systems, we have to specify a couple of other things.

First, we will use natural logarithm $\ln$ in entropic quantities, as is customary in settings of continuous alphabets, resulting in the entropies be counted in units of *nats*. For a density operator α, the von Neumann entropy is defined as [38]

$$S(\alpha) := -\operatorname{Tr}\alpha\ln\alpha. \tag{8}$$

For two density operators α and β such that $supp(\alpha) \subseteq supp(\beta)$, the quantum relative entropy of α with respect to β is defined as

$$D(\alpha\|\beta) := \operatorname{Tr}\alpha(\ln\alpha - \ln\beta); \tag{9}$$

otherwise, $D(\alpha\|\beta) := \infty$.

Second, we have to introduce a Hamiltonian operator to constrain the states of a system in order to avoid unphysical results.

A Hamiltonian H_A is a densely defined self-adjoint operator on the Hilbert space of a quantum system A, that is bounded from below. One way of defining such an operator is to let $\{|e_j\rangle\}$ be an orthonormal basis for the Hilbert space under consideration (e.g. Fock basis), and $\{a_j\}$, a sequence of real numbers bounded from below. Then,

$$H_A|\psi\rangle := \sum_{j=1}^{\infty} a_j|e_j\rangle\langle e_j|\psi\rangle, \tag{10}$$

defines H_A on the dense subspace $\mathcal{I} = \Big\{|\psi\rangle : \sum_{j=1}^{\infty} a_j^2|\langle e_j|\psi\rangle|^2 < +\infty\Big\}$, with $\{a_j\}$ the eigenvalues corresponding to the eigenvectors $\{|e_j\rangle\}$. All Hamiltonians with discrete spectrum arise in this way.

For an arbitrary state ρ, the expectation of H_A is given by

$$\operatorname{Tr}\rho H_A = \sum_{j=1}^{\infty} a_j \langle e_j|\rho|e_j\rangle. \tag{11}$$

The n-th extension H_{A^n} of the energy observable H_A to the system $A^n = A^{\otimes n}$ is defined in an i.i.d. fashion as follows:

$$\begin{aligned} H_{A^n} := {} & H_A \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\ & + \mathbb{1} \otimes H_A \otimes \cdots \otimes \mathbb{1} \\ & + \ldots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes H_A. \end{aligned} \tag{12}$$

A. Gaussian States

We recall here some basic facts about Gaussian states that are at the core of our treatment, which also serves the purpose of fixing the notations used in following sections. The canonical observables $\hat{\boldsymbol{r}} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^\top$ describe a Bosonic system of N harmonic modes in a Hilbert space $X = \bigotimes_{k=1}^N X_k$. On such a system, we consider by default a quadratic Hamiltonian, whose most general form is

$$H_X = \hat{\boldsymbol{r}}_X \boldsymbol{\Omega}_X \hat{\boldsymbol{r}}_X^\top, \tag{13}$$

where $\boldsymbol{\Omega}_X$ is positive symmetric matrix, assumed for the sake of simplicity to have a unique N-fold degenerate eigenvalue ω^X. Hence in normal form, $H_X = \omega^X \sum_{j=1}^N (\hat{q}_j^2 + \hat{p}_j^2)/2$.

Hereafter we denote vectors (resp. matrices) by lower (resp. upper) case bold symbols. The Heisenberg canonical commutation relations satisfied by the canonical observables can be compactly represented as

$$[\hat{r}_j, \hat{r}_k] = i\Sigma_{jk}, \quad \forall j,k \in \{1,\ldots,2N\}, \tag{14}$$

with

$$\boldsymbol{\Sigma} := \bigoplus_1^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{15}$$

and $\hat{r}_{2k-1} = \hat{q}_k$, $\hat{r}_{2k} = \hat{p}_k$. For any density operator ρ acting on X, the vector mean (or first moment) is the vector $\boldsymbol{d} \in \mathbb{R}^{2N}$, whose components are given by

$$d_k := \operatorname{Tr}\rho\hat{r}_k. \tag{16}$$

The $2N \times 2N$ covariance matrix (CM) $\boldsymbol{V}$ is given by

$$V_{jk} := \operatorname{Tr}\rho\{(\hat{r}_j - d_j)(\hat{r}_k - d_k) + (\hat{r}_k - d_k)(\hat{r}_j - d_j)\}, \tag{17}$$

which is real, symmetric and positive definite. Furthermore, for the CM to correspond to a bona fide quantum state it has to satisfy the following Heisenberg-Robertson uncertainty relation

$$\boldsymbol{V} + i\boldsymbol{\Sigma} \geq 0. \tag{18}$$

Conversely, if the uncertainty relation is satisfied, there exists a quantum state with CM $\boldsymbol{V}$, in fact a Gaussian state ρ, that is uniquely defined by its associated Gaussian characteristic function

$$\chi_\rho(\boldsymbol{\zeta}) = \exp\left(-i(\boldsymbol{\Sigma d})^\top \boldsymbol{\zeta} - \frac{1}{4}\boldsymbol{\zeta}^\top \boldsymbol{\Sigma V \Sigma}^\top \boldsymbol{\zeta}\right), \tag{19}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^{2N}$. Recall that the (zero-ordered) characteristic function is defined as

$$\chi_\rho(\boldsymbol{\zeta}) := \operatorname{Tr}(\rho W_{\boldsymbol{\zeta}}), \tag{20}$$

with the Weyl displacement operator given by

$$W_{\boldsymbol{\zeta}} := \exp\left(-i\hat{\boldsymbol{r}}^\top \boldsymbol{\Sigma\zeta}\right). \tag{21}$$

Thus, Gaussian states are completely characterized by $\boldsymbol{d}$ and $\boldsymbol{V}$.

The von Neumann entropy (8) of an N-mode Gaussian state ρ can be evaluated through its covariance matrix as

$$S(\rho) = S(\boldsymbol{V}) = \sum_{i=1}^N g(\nu_i), \tag{22}$$

where $\nu_1, \ldots, \nu_N$ are the symplectic eigenvalues of $\mathbf{V}$. Note that for Gaussian states, the entropy is a function entirely of the CM, and so we slightly abuse notation writing $S(\boldsymbol{V})$. Here the function g is defined by

$$g(x) := \left(x + \frac{1}{2}\right)\ln\left(x + \frac{1}{2}\right) - \left(x - \frac{1}{2}\right)\ln\left(x - \frac{1}{2}\right), \tag{23}$$

and as such $g(x)$ is an increasing and concave function.

B. Gaussian Unitaries

Since we shall consider Gaussian unitaries in place of W in Fig. 1, let us recall basic notions about them.

Consider N Bosonic modes. A Gaussian unitary on them $\exp(-iH)$ with H as in Eq. (13), can be simply described by an affine map

$$(\boldsymbol{S}, \boldsymbol{\zeta}) : \hat{\boldsymbol{r}} \to \boldsymbol{S}\hat{\boldsymbol{r}} + \boldsymbol{\zeta}, \tag{24}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^{2\ N}$ and $\boldsymbol{S} \in Sp(2N, \mathbb{R})$ because the transformation must preserve the commutation relations (14). Clearly the eigenvalues $\boldsymbol{r}$ of the quadrature operators $\hat{\boldsymbol{r}}$ must follow the same rule, i.e.,

$$(\boldsymbol{S}, \boldsymbol{\zeta}) : \boldsymbol{r} \to \boldsymbol{S}\boldsymbol{r} + \boldsymbol{\zeta}. \tag{25}$$

Thus, a Gaussian unitary is equivalent to an affine symplectic map $(\boldsymbol{S}, \boldsymbol{\zeta})$ acting on the phase space, and can be denoted by $\boldsymbol{U}_{\boldsymbol{S},\boldsymbol{\zeta}}$. In particular, we can write

$$\boldsymbol{U}_{\boldsymbol{S},\boldsymbol{\zeta}} = W_{\boldsymbol{\zeta}} \boldsymbol{U}_{\boldsymbol{S}}, \tag{26}$$

where the canonical unitary $\boldsymbol{U}_{\boldsymbol{S}}$ corresponds to a linear symplectic map $\boldsymbol{r} \to \boldsymbol{S}\boldsymbol{r}$, and the Weyl operator $W_{\boldsymbol{\zeta}}$ to a phase-space translation $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{\zeta}$.

In terms of the statistical moments, $\boldsymbol{d}$ and $\boldsymbol{V}$, the action of $\boldsymbol{U}_{\boldsymbol{S},\boldsymbol{\zeta}}$ is characterized by the following transformations

$$\boldsymbol{d} \to \boldsymbol{S}\boldsymbol{d} + \boldsymbol{\zeta}, \quad \boldsymbol{V} \to \boldsymbol{S}\boldsymbol{V}\boldsymbol{S}^\top. \tag{27}$$

Therefore, the action of a Gaussian unitary $\boldsymbol{U}_{\boldsymbol{S},\boldsymbol{\zeta}}$ over a Gaussian state $\rho(\boldsymbol{d},\boldsymbol{V})$ will be completely described by Eq. (27).

Note that the above arguments also apply if we replace the vector of quadrature operators $\hat{\boldsymbol{r}}$ by the vector of ladder operators (also known as annihilation and creation operators) $\hat{\boldsymbol{v}} = (\hat{a}_1, \hat{a}_1^\dagger, \cdots, \hat{a}_n, \hat{a}_n^\dagger)^\top$, where

$$\hat{a}_j = \frac{\hat{q}_j + i\hat{p}_j}{\sqrt{2}}. \tag{28}$$

In such a case however, it will be $\boldsymbol{S} \in Sp(2N,\mathbb{C})$. Let us now focus on two-mode Gaussian unitaries. Consider $\hat{\boldsymbol{v}} = (\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger)^\top$ with

$$\hat{a} = \frac{\hat{q}_a + i\hat{p}_a}{\sqrt{2}}, \quad \hat{b} = \frac{\hat{q}_b + i\hat{p}_b}{\sqrt{2}}. \tag{29}$$

Then, the canonical unitary of Eq. (26), named here $\boldsymbol{U}_{ab}$, satisfies

$$\boldsymbol{U}_{ab}\,\hat{\boldsymbol{v}}\,\boldsymbol{U}_{ab}^\dagger = \boldsymbol{S}\cdot\hat{\boldsymbol{v}}, \tag{30}$$

with $\boldsymbol{S} \in Sp(4,\mathbb{C})$. Define

$$q := |S_{11}|^2 - |S_{12}|^2, \tag{31}$$

where S_{11} and S_{12} are matrix elements of $\boldsymbol{S}$. In [4, App. A], it is shown that for $q > 0,\ q \neq 1$

$$\boldsymbol{U}_{ab} = (\boldsymbol{S}_a \otimes \boldsymbol{S}_b)\boldsymbol{U}_{ab}^{(q)}\big(I_a \otimes \boldsymbol{S}_b'\big), \tag{32}$$

where $\boldsymbol{S}_a$, $\boldsymbol{S}_b$ and $\boldsymbol{S}_b'$ denote (generally different) one-mode squeezing transformations. For $q \in (0,1)$, $\boldsymbol{U}_{ab}^{(q)}$ is characterized by the symplectic matrix

$$\boldsymbol{S}_{ab}^{(q)} = \begin{pmatrix} \sqrt{q} & 0 & -\sqrt{1-q} & 0 \\ 0 & \sqrt{q} & 0 & -\sqrt{1-q} \\ \sqrt{1-q} & 0 & \sqrt{q} & 0 \\ 0 & \sqrt{1-q} & 0 & \sqrt{q} \end{pmatrix}, \tag{33}$$

while for $q > 1$, by

$$\boldsymbol{S}_{ab}^{(q)} = \begin{pmatrix} \sqrt{q} & 0 & 0 & -\sqrt{q-1} \\ 0 & \sqrt{q} & -\sqrt{q-1} & 0 \\ 0 & -\sqrt{q-1} & \sqrt{q} & 0 \\ -\sqrt{q-1} & 0 & 0 & \sqrt{q} \end{pmatrix}. \tag{34}$$

The case $q < 0$ can be traced back to the case $q > 0$ by the following argument. Consider the transformation SWAP_{ab} swapping (exchanging) the two modes, defined by

$$\begin{aligned} \mathrm{SWAP}_{ab} &= \mathrm{SWAP}_{ab}^\dagger, \\ \mathrm{SWAP}_{ab}\,\hat{a}\,\mathrm{SWAP}_{ab}^\dagger &= \hat{b}, \\ \mathrm{SWAP}_{ab}\,\hat{b}\,\mathrm{SWAP}_{ab}^\dagger &= \hat{a}. \end{aligned} \tag{35}$$

Therefore, one gets the following relation:

$$\mathbf{SWAP}_{ab}\boldsymbol{U}_{ab}\hat{\boldsymbol{v}}\boldsymbol{U}_{ab}^\dagger\mathbf{SWAP}_{ab} = \widetilde{\boldsymbol{S}}\cdot\hat{\boldsymbol{v}}, \tag{36}$$

where $\widetilde{\boldsymbol{S}}$ is a 4×4 matrix obtained by shifting by 2 the columns of the symplectic matrix $\boldsymbol{S}$ describing the unitary $\boldsymbol{U}_{ab}$. In other words,

$$\widetilde{S}_{ij} = S_{i,j\oplus2}, \tag{37}$$

where $\oplus$ denotes the sum modulo 4. This, in turn, causes a parameter change $q \mapsto 1-q > 1$ so that (32) has to be rewritten like

$$\mathbf{SWAP}_{ab}\boldsymbol{U}_{ab} = (\boldsymbol{S}_a \otimes \boldsymbol{S}_b)\boldsymbol{U}_{ab}^{(1-q)}\big(I_a \otimes \boldsymbol{S}_b'\big), \tag{38}$$

where $\boldsymbol{U}_{ab}^{(1-q)}$ will be characterized by a symplectic matrix $\boldsymbol{S}_{ab}^{(1-q)}$ of the same form of (34) (with the substitution $q \mapsto 1-q$). Consequently, for $q < 0$,

$$\boldsymbol{U}_{ab} = \mathbf{SWAP}_{ab}(\boldsymbol{S}_a \otimes \boldsymbol{S}_b)\boldsymbol{U}_{ab}^{(1-q)}\big(I_a \otimes \boldsymbol{S}_b'\big). \tag{39}$$

C. Gaussian Quantum Channels

Here, in the perspective of dealing with channels (3), (4), (5) that would be mostly Gaussian, we recall the definition of this latter kind of quantum channels. Moreover we present Lemma 1 showing how separate energy constraints on systems A and E reflect onto system B of Fig. 1.

A Bosonic Gaussian channel (BGC) $\mathcal{N}^{A\to B}$ is a linear completely positive and trace preserving map defined on $\mathcal{T}(A)$ and taking values in $\mathcal{T}(B)$, that maps every Gaussian state to a Gaussian state. As Gaussian states span all states and are completely characterized by their first and second moments, the BGC $\mathcal{N}^{A\to B}$ can be completely characterized by the rule of transformations on the vector mean and the covariance matrix. On the level of vector mean and covariance matrices, the action of $\mathcal{N}^{A\to B}$ is as follows:

$$\begin{aligned} \boldsymbol{d}_A &\mapsto \boldsymbol{d}_B = \boldsymbol{X}\boldsymbol{d}_A + \boldsymbol{d}_E, \\ \boldsymbol{V}_A &\mapsto \boldsymbol{V}_B = \boldsymbol{X}\boldsymbol{V}_A\boldsymbol{X}^\top + \boldsymbol{Y}, \end{aligned} \tag{40}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are real matrices with $\boldsymbol{Y} = \boldsymbol{Y}^\top$ and $\boldsymbol{Y} \geq 0$. For this transformation to represent a quantum channel, we must have

$$\boldsymbol{Y} + i\boldsymbol{\Sigma} \geq i\boldsymbol{X}\boldsymbol{\Sigma}\boldsymbol{X}^\top. \tag{41}$$

In particular, when $\boldsymbol{Y} = 0$, the channel $\mathcal{N}^{A\to B}$ represents a unitary evolution of the system and from Eq. (18), it follows that $\boldsymbol{X}$ is a symplectic matrix. Thus, the action of a Gaussian unitary $U^{A\to B}$ on the state ρ^A with N_A modes can be described by a symplectic matrix of size $2N_A \times 2N_A$ as follows:

$$\rho^B = U\rho^A U^\dagger \leftrightarrow \boldsymbol{V}_B = \boldsymbol{S}\boldsymbol{V}_A\boldsymbol{S}^\top. \tag{42}$$

It is furthermore well-known that a quantum channel can be seen as part of a unitary evolution on a larger system whose ancillary parts are not under our control. Actually, every BGC acting on N_A modes can be represented by a unitary operation $U^{AE\to BF}$ on the system and a minimal environment of N_E modes, where $N_E \leq 2N_A$. This unitary interaction, extending the argument of Subsection II-B) to multimodes, can be described by a symplectic matrix $\boldsymbol{S}$, written in block form as follows:

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{N} \\ \boldsymbol{O} & \boldsymbol{P} \end{pmatrix}. \tag{43}$$

When the input state in the environment is $\boldsymbol{V}_E$, the effective channel $\mathcal{N}_{\boldsymbol{V}_E}^{A\to B}$ can be described as

$$\boldsymbol{V}_A \mapsto \boldsymbol{V}_B = \boldsymbol{M}\boldsymbol{V}_A\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{V}_E\boldsymbol{N}^\top. \tag{44}$$

In turn, the complementary channel $\widetilde{\mathcal{N}}_{\boldsymbol{V}_E}^{A\to F}$ acts on the CM as

$$\boldsymbol{V}_A \mapsto \boldsymbol{V}_F = \boldsymbol{O}\boldsymbol{V}_A\boldsymbol{O}^\top + \boldsymbol{P}\boldsymbol{V}_E\boldsymbol{P}^\top. \tag{45}$$

Lemma 1: Let $\mathcal{N}^{AE\to B}$ be a Gaussian channel from system AE to system B with input Gaussian states subject to the conditions $\operatorname{Tr}\rho H_A \leq P_A$ and $\operatorname{Tr}\eta H_E \leq P_E$, for density operators ρ and η on systems A and E, respectively. Then, there exists a quadratic Hamiltonian H_B on system B such that

$$\operatorname{Tr}\mathcal{N}(\rho\otimes\eta)H_B \leq 2P_A + 2P_E,$$

and

$$\operatorname{Tr} e^{-\beta H_B} < \infty,\ \forall\beta > 0. \tag{46}$$

Furthermore, it holds

$$\sup_{\eta:\operatorname{Tr}(\eta H_E)\leq P_E}\ \sup_{\rho:\operatorname{Tr}(\rho(H_A))\leq P_A} S(\mathcal{N}(\rho\otimes\eta)) < \infty. \tag{47}$$

Proof: Let us generically consider each system A, E, B to be composed of N modes, and recall from Eq. (13), that

$$H_A = \hat{\boldsymbol{r}}_A\boldsymbol{\Omega}_A\hat{\boldsymbol{r}}_A^\top, \tag{48}$$

as well as

$$H_E = \hat{\boldsymbol{r}}_E\boldsymbol{\Omega}_E\hat{\boldsymbol{r}}_E^\top, \tag{49}$$

to be quadratic Hamiltonians, where $\boldsymbol{\Omega}_A$ and $\boldsymbol{\Omega}_E$ are positive matrices with eigenvalues ω^A and ω^E.

On the system A (resp. E), for a given state ρ (resp. η) with covariance matrix $\boldsymbol{V}_\rho$ (resp. $\boldsymbol{V}_\eta$) the constrained energy is given by $\operatorname{Tr}\rho H_A = \operatorname{Tr}\boldsymbol{\Omega}_A\boldsymbol{V}_\rho + \boldsymbol{d}_A\boldsymbol{\Omega}_A\boldsymbol{d}_A^\top \leq P_A$, (resp. $\operatorname{Tr}\rho H_E = \operatorname{Tr}\boldsymbol{\Omega}_E\boldsymbol{V}_\eta + \boldsymbol{d}_E\boldsymbol{\Omega}_E\boldsymbol{d}_E^\top \leq P_E$). Let us define

$$\begin{aligned} H_{B,\eta} := c\Big(&\hat{\boldsymbol{r}}_B\hat{\boldsymbol{r}}_B^\top - (\operatorname{Tr}\boldsymbol{V}_\eta\boldsymbol{N}^\top\boldsymbol{N})\mathbb{1}_B \\ &- \boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top\mathbb{1}_B\Big), \end{aligned} \tag{50}$$

where c is a positive real constant. We know that $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{\Omega_A}$ and $\boldsymbol{\Omega}_E$ are finite dimensional matrices. Therefore, it is possible to choose constants $c_A > 0$ and $c_E > 0$ such that $\boldsymbol{M}^\top\boldsymbol{M} \leq c_A\boldsymbol{\Omega}_A$ and $\boldsymbol{N}^\top\boldsymbol{N} \leq c_E\boldsymbol{\Omega}_E$. As a consequence,

$$\begin{aligned} &\operatorname{Tr}\boldsymbol{V}_\eta\boldsymbol{N}^\top\boldsymbol{N} + \boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top \\ &\leq c_E\operatorname{Tr}\boldsymbol{\Omega}_E\boldsymbol{V}_\eta + c_E\boldsymbol{d}_\eta\boldsymbol{\Omega}_E\boldsymbol{d}_\eta^\top \\ &= c_E\operatorname{Tr}\eta H_E \\ &\leq c_E P_E, \end{aligned} \tag{51}$$

hence

$$H_{B,\eta} \geq c(\hat{\boldsymbol{r}}_B\hat{\boldsymbol{r}}_B^\top - c_E P_E\mathbb{1}_B). \tag{52}$$

In other words, the eigenvalues of $H_{B,\eta}$ are bounded from below. Therefore, we have

$$\begin{aligned} \operatorname{Tr}\exp(-\beta H_{B,\eta}) &\leq \operatorname{Tr}\exp(-\beta c\hat{\boldsymbol{r}}_B\hat{\boldsymbol{r}}_B^\top)\exp(\beta c c_E P_E) \\ &< \infty. \end{aligned} \tag{53}$$

On the other hand, we have

$$\begin{aligned} \operatorname{Tr}\rho\mathcal{N}_\eta^*(H_{B,\eta}) &= \operatorname{Tr}\mathcal{N}_\eta(\rho)H_{B,\eta} \\ &= c\operatorname{Tr}\mathcal{N}_\eta(\rho)\hat{\boldsymbol{r}}_B\hat{\boldsymbol{r}}_B^\top - c\operatorname{Tr}\boldsymbol{V}_\eta\boldsymbol{N}^\top\boldsymbol{N} \\ &\quad - c\boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top, \end{aligned} \tag{54}$$

hence

$$\begin{aligned} &\operatorname{Tr}\rho\mathcal{N}_\eta^*(H_{B,\eta}) \\ &= c\operatorname{Tr}(\boldsymbol{M}^\top\boldsymbol{V}_\rho\boldsymbol{M} + \boldsymbol{N}^\top\boldsymbol{V}_\eta\boldsymbol{N}) \\ &\quad + c\Big(\boldsymbol{d}_\rho\boldsymbol{M}^\top + \boldsymbol{d}_\eta\boldsymbol{N}^\top\Big)\Big(\boldsymbol{d}_\rho\boldsymbol{M}^\top + \boldsymbol{d}_\eta\boldsymbol{N}^\top\Big)^\top \\ &\quad - c\operatorname{Tr}\boldsymbol{V}_\eta\boldsymbol{N}^\top\boldsymbol{N} - c\boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top \\ &= c\operatorname{Tr}\boldsymbol{M}^\top\boldsymbol{V}_\rho\boldsymbol{M} \\ &\quad + c\Big(\boldsymbol{d}_\rho\boldsymbol{M}^\top + \boldsymbol{d}_\eta\boldsymbol{N}^\top\Big)\Big(\boldsymbol{d}_\rho\boldsymbol{M}^\top + \boldsymbol{d}_\eta\boldsymbol{N}^\top\Big)^\top \\ &\quad - c\boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top. \end{aligned} \tag{55}$$

From the triangle inequality, we have

$$\begin{aligned} &\Big(\boldsymbol{d}_\rho\boldsymbol{M}^\top + \boldsymbol{d}_\eta\boldsymbol{N}^\top\Big)\Big(\boldsymbol{d}_\rho\boldsymbol{M}^\top + \boldsymbol{d}_\eta\boldsymbol{N}^\top\Big)^\top \\ &\leq 2\boldsymbol{d}_\rho\boldsymbol{M}^\top\boldsymbol{M}\boldsymbol{d}_\rho^\top + 2\boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top. \end{aligned} \tag{56}$$

From the above inequality, one gets

$$\begin{aligned} &\operatorname{Tr}\rho\mathcal{N}_\eta^*(H_{B,\eta}) \\ &\leq c\operatorname{Tr}\rho\,\hat{\boldsymbol{r}}_A\boldsymbol{M}^\top\boldsymbol{M}\hat{\boldsymbol{r}}_A^\top + 2c\boldsymbol{d}_\rho\boldsymbol{M}^\top\boldsymbol{M}\boldsymbol{d}_\rho^\top \\ &\quad + 2c\boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top - c\boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top \\ &\leq c\operatorname{Tr}\rho\,\hat{\boldsymbol{r}}_A\boldsymbol{M}^\top\boldsymbol{M}\hat{\boldsymbol{r}}_A^\top + 2c\boldsymbol{d}_\rho\boldsymbol{M}^\top\boldsymbol{M}\boldsymbol{d}_\rho^\top \\ &\quad + c\boldsymbol{d}_\eta\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{d}_\eta^\top \\ &\leq cc_A\operatorname{Tr}\rho\,\hat{\boldsymbol{r}}_A\boldsymbol{\Omega}_A\hat{\boldsymbol{r}}_A^\top + 2cc_A\boldsymbol{d}_\rho\boldsymbol{\Omega}_A\boldsymbol{d}_\rho^\top \\ &\quad + cc_E\boldsymbol{d}_\eta\boldsymbol{\Omega}_E\boldsymbol{d}_\eta^\top \\ &\leq \operatorname{Tr}\rho\,\hat{\boldsymbol{r}}_A\boldsymbol{\Omega}_A\hat{\boldsymbol{r}}_A^\top + 2\boldsymbol{d}_\rho\boldsymbol{\Omega}_A\boldsymbol{d}_\rho^\top + \boldsymbol{d}_\eta\boldsymbol{\Omega}_E\boldsymbol{d}_\eta^\top \\ &\leq \operatorname{Tr}\rho H_A + \boldsymbol{d}_\rho\boldsymbol{\Omega}_A\boldsymbol{d}_\rho^\top + \boldsymbol{d}_\eta\boldsymbol{\Omega}_E\boldsymbol{d}_\eta^\top \\ &\leq 2P_A + P_E, \end{aligned} \tag{57}$$

where c is selected such that $cc_A, cc_E \leq 1$. Now, set

$$H_B := c\,\hat{\boldsymbol{r}}_B\hat{\boldsymbol{r}}_B^\top, \tag{58}$$

which evidently is a positive self-adjoint operator independent of η and ρ. It trivially satisfies

$$\begin{aligned} \operatorname{Tr}\mathcal{N}(\rho\otimes\eta)H_B &= \operatorname{Tr}\mathcal{N}_\eta(\rho)H_B \\ &= \operatorname{Tr}\rho\mathcal{N}_\eta^*(H_B) \\ &= \operatorname{Tr}\rho\mathcal{N}_\eta^*(H_B - cc_E P_E\mathbb{1}) \\ &\quad + \operatorname{Tr}\rho\mathcal{N}_\eta^*(cc_E P_E\mathbb{1}), \end{aligned} \tag{59}$$

and thanks to Eq. (57), we have

$$\begin{aligned} \operatorname{Tr}\mathcal{N}(\rho\otimes\eta)H_B &\leq \operatorname{Tr}\rho\mathcal{N}_\eta^*(H_{B,\eta}) + cc_E P_E \\ &\leq 2P_A + P_E + cc_E P_E \\ &\leq 2P_A + 2P_E, \end{aligned} \tag{60}$$

concluding the proof. ■

III. Quantum Communication

In this section, we discuss the model of quantum communication with environment-assistance. We first focus on the unconstrained quantum capacity, for which we refer to isometries giving rise to BGC for each choice of Gaussian initial

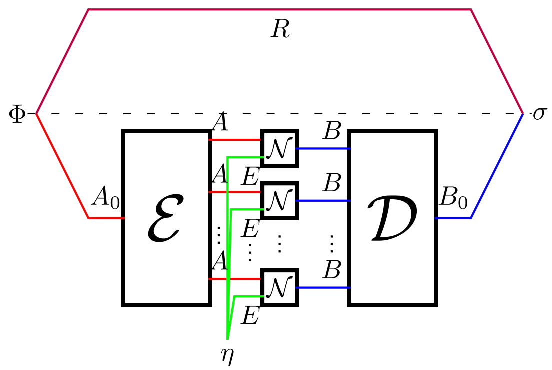

Fig. 2. Schematic of the protocol for transmitting quantum information with passive assistance from the environment; $\mathcal{E}$ and $\mathcal{D}$ are the encoding and decoding maps respectively.

environment state, and then move on to energy-constrained quantum capacities.

Referring to Fig. 2, given a Gaussian isometry $W : AE \to BF$, to send quantum information down the channel $\mathcal{N}_\eta(\rho) = \operatorname{Tr}_F W(\rho \otimes \eta)W^\dagger$ from Alice to Bob, we need an encoding CPTP map $\mathcal{E} : \mathcal{T}(A_0) \to \mathcal{T}(A^n)$ and a decoding CPTP map $\mathcal{D} : \mathcal{T}(B^n) \to \mathcal{T}(B_0)$, where the number of qubits of A_0 is equal to that of B_0. The output, upon inputting a maximally entangled state Φ^{RA_0} with R being an inaccessible reference system, reads $\sigma^{RB_0} = \mathcal{D}\big(\mathcal{N}^{\otimes n}\big(\mathcal{E}\big(\Phi^{RA_0}\big) \otimes \eta^{E^n}\big)\big)$.

Definition 2: A passive environment-assisted quantum code of block length n is given by a triple $\big(\mathcal{E}^{A_0 \to A^n}, \eta^{E^n}, \mathcal{D}^{B^n \to B_0}\big)$ of an encoding map, a helper state and a decoding map. Its fidelity is given by $\operatorname{Tr} \Phi^{RA_0}\sigma^{RB_0}$ and its rate by the number of qubits of A_0 divided by n.

A rate $\mathtt{R}$ is called achievable if there are codes for all block lengths n with fidelity converging to 1 and rate converging to $\mathtt{R}$. The *passive environment-assisted quantum capacity* of W, denoted by $Q_H(W)$, is the supremum of all achievable rates.

If the helper is restricted to fully separable states η^{E^n}, i.e. convex combinations of tensor products $\eta^{E^n} = \eta^{E^1} \otimes \ldots \otimes \eta^{E^n}$, the supremum of all achievable rates is called *separable passive environment-assisted quantum capacity* and denoted by $Q_{H\otimes}(W)$.

If in addition the helper is restricted to Gaussian states, we get the *Gaussian separable passive environment-assisted quantum capacity*, which we denote $Q_{GH\otimes}(W)$.

Theorem 3: For a Gaussian isometry $W : AE \to BF$, the passive environment-assisted quantum capacity is given by

$$Q_H(W) = \sup_n \max_{\eta^{(n)}} \frac{1}{n} Q\Big(\mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big) = \sup_n \max_{\rho^{(n)}, \eta^{(n)}} \frac{1}{n} I_c\Big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big), \quad (61)$$

where the maximization is over states $\rho^{(n)}$ on A^n and states $\eta^{(n)}$ on E^n.

Similarly, the capacity with separable helper is given by the same formula,

$$Q_{H\otimes}(W) = \sup_n \max_{\eta_1 \otimes \ldots \otimes \eta_n} \frac{1}{n} Q(\mathcal{N}_{\eta_1} \otimes \cdots \otimes \mathcal{N}_{\eta_n}) = \sup_n \max_{\rho^{(n)}, \eta^{(n)}} \frac{1}{n} I_c\Big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big), \quad (62)$$

but now varying only over product states $\eta^{(n)} = \eta_1 \otimes \ldots \otimes \eta_n$. Consequently,

$$Q_H(W) = \lim_{n \to \infty} \frac{1}{n} Q_{H\otimes}(W^{\otimes n}). \quad (63)$$

Proof: It is known that the coherent information for nontrivial Gaussian channels without constrained energy is finite [3]. However, relations (61) and (62) without energy constraint may be infinite. To guarantee their finiteness, one has to exploit energy constraints together with subadditivity and concavity of von Neumann entropy.

The direct part, i.e. the "$\geq$" inequality, follows from the Lloyd-Shor-Devetak theorem applied to the channel $(\mathcal{N})_{\eta^{(n)}}$ (to be precise, to many copies of this block channel, so that the i.i.d. theorems apply, cf. [31]).

For the converse part, i.e. "$\leq$", the proof is like [16], which is based on the argument of [1], [25], [26]. More specifically, given a code of block length n, whatever is the environment state $\eta^{(n)}$, the upper bound on (61) is found by applying the Fannes inequality and the convexity of the coherent information. ■

A. Universal (Anti-)Degradability Properties

One of the main problems in quantum information theory is to express the quantum capacity by a single-letter formula. This can be done when the channel possesses the (anti-)degradability property, which guarantees the additivity of the coherent information [7]. Here we want to understand, for a given two-mode Gaussian unitary, whether or not this property can hold true irrespective of the environment state.

Recall that degradability of $\mathcal{N}_{\eta_E}^{A \to B}$ is defined by the existence of a CPTP map $\Gamma^{B \to F}$ such that

$$\widetilde{\mathcal{N}}_{\eta_E}^{A \to F} = \Gamma^{B \to F} \circ \mathcal{N}_{\eta_E}^{A \to B}. \quad (64)$$

Analogously, anti-degradability is defined by the existence of a map $\bar{\Gamma}^{F \to B}$ such that

$$\bar{\Gamma}^{F \to B} \circ \widetilde{\mathcal{N}}_{\eta_E}^{A \to F} = \mathcal{N}_{\eta_E}^{A \to B}. \quad (65)$$

Remark 4: By looking at the discussion in Subsection II-B, we can see that any two-mode unitary $U^{(q)}$ with $q \geq 1/2$ is degradable with respect to all Gaussian environment pure states; we say that the unitary is *Gaussian universally degradable.*

This comes from the fact that for the Gaussian quantum channel $\mathcal{N}_{\boldsymbol{V}_E, q}^{A \to B}$ we can find the required channel $\Gamma^{B \to F}$ in Eq. (64) as $\mathcal{N}_{\boldsymbol{V}_E, \frac{2q-1}{q}}^{F \to B}$, because

$$\widetilde{\mathcal{N}}_{\boldsymbol{V}_E, q}^{A \to F} = \widetilde{\mathcal{N}}_{\boldsymbol{V}_E, \frac{2q-1}{q}}^{B \to F} \circ \mathcal{N}_{\boldsymbol{V}_E, q}^{A \to B}. \quad (66)$$

Remark 5: By looking at the discussion in Subsection II-B, we can see that any two-mode unitary $U^{(q)}$ with $0 \leq q \leq 1/2$ is anti-degradable with respect to all Gaussian environment pure states; we say that the unitary is *Gaussian universally anti-degradable*.

This comes from the fact that for the Gaussian quantum channel $\mathcal{N}_{\boldsymbol{V}_E, q}^{A \to B}$ we can find the required channel $\bar{\Gamma}^{F \to B}$ in

Eq. (65) as $\widetilde{\mathcal{N}}^{F\to B}_{\boldsymbol{V}_E,\frac{1-2q}{1-q}}$, because

$$\mathcal{N}^{A\to B}_{\boldsymbol{V}_E,q} = \widetilde{\mathcal{N}}^{F\to B}_{\boldsymbol{V}_E,\frac{1-2q}{1-q}} \circ \widetilde{\mathcal{N}}^{A\to F}_{\boldsymbol{V}_E,q}. \tag{67}$$

Definition 6: A two-mode Gaussian unitary U is said to be *universally degradable* (resp. *universally anti-degradable*) if Eq. (64) (resp. (65)) holds true for all environment states η_E.

Theorem 7: Any two-mode Gaussian unitary $U^{(q)}$ is neither universally degradable, nor universally anti-degradable, unless $q = 1$.

The proof of this theorem, which we give in Appendix A, is obtained by assuming the existence of a quantum channel Γ satisfying the degradability condition (64) and then showing that this leads to a contradiction. In particular, for $q \le 1/2$ the claim follows from the fact that the channel is Gaussian universally anti-degradable, but has positive coherent information, and hence cannot be anti-degradable, for some non-Gaussian environment states [19].

Corollary 8: The two-mode Gaussian unitary $U^{(q)} : AE \to BF$ with $q \ge 1/2$ is Gaussian universally degradable, and hence its Gaussian separable passive environment-assisted quantum capacity is given by the single-letter formula

$$Q_{GH\otimes}(U^{(q)}) = \max_{\eta_G} \sup_{\rho_G} I_c(\rho_G; \mathcal{N}_\eta), \tag{68}$$

where the optimization can be restricted to Gaussian input states ρ_G (cf. [12, Thm. 12.38]). Note that, for each fixed η_G, finding a local maximum of I_c corresponds to finding the global one, thanks to the concavity of coherent information.

For $q \le 1/2$, the two-mode Gaussian unitary $U^{(q)} : AE \to BF$ is Gaussian universally anti-degradable, and hence its Gaussian separable passive environment-assisted quantum capacity vanishes, $Q_{GH\otimes}(U^{(q)}) = 0$. ■

Armed with this corollary, we can now proceed to calculate the Gaussian separable passive environment-assisted quantum capacity of the two-mode unitaries $U^{(q)}$. Note that for each Gaussian environment state η_G, the resulting channel $\mathcal{N}_\eta$ is an OMG, a one-mode Gaussian channel. Their complete classification is given in [13]. In particular, when $\eta = |0\rangle\langle 0|$ is the vacuum state, $U^{(q)}$ gives rise to an attenuator channel for $q < 1$, and an amplifier channel for $q > 1$; for $q = 1$, $\mathcal{N}_{|0\rangle\langle 0|}$ is the identity.

For an OMG channel described by Eq. (40), the parameters that characterize it are

$$x := \sqrt{\det \boldsymbol{X}}, \quad y := \det \boldsymbol{Y}. \tag{69}$$

Furthermore, we define another parameter dependent on these two, $K := \frac{1}{2}(y - |1 - x|)$.

For OMG channels, whenever the coherent information is non-zero, the supremum over all Gaussian input states is achieved for infinite input power, $P_A \to \infty$. It is known from [3] that the optimised coherent information (over all Gaussian input states) is given by

$$\begin{aligned}\sup_{\rho_G} I_c(\rho_G; \mathcal{N}) = &\frac{K}{|1-x|} \ln \frac{K}{|1-x|} \\ &- \frac{K + |1-x|}{|1-x|} \ln \frac{K + |1-x|}{|1-x|} \\ &+ \ln \frac{x}{|1-x|}.\end{aligned} \tag{70}$$

For $0 \le q \le 1$, $U^{(q)}$ with the symplectic matrix (33) describes a beam splitter with transmissivity q and is denoted hereafter by $B(q)$. Considering $1/2 \le q < 1$, then from Corollary 8 we have

$$Q_{GH\otimes}(B(q)) = \max_{\boldsymbol{V}_E} \sup_{\boldsymbol{V}_A} I_c(\boldsymbol{V}_A; B(q)), \tag{71}$$

where the maximization over environment states can be restricted to pure one-mode states given by the covariance matrix

$$\boldsymbol{V}_E = \begin{pmatrix} \cosh(2s)(1+\cos\theta) & \sin\theta \sinh(2s) \\ \sin\theta\sinh(2s) & \cosh(2s)(1-\cos\theta) \end{pmatrix}, \tag{72}$$

with $s \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. Eqs. (33) and (69) yield $x = q$ and $y = 1 - q$ for all one-mode squeezed input environment $\boldsymbol{V}_E$. Invoking Eq. (70), we get

$$Q_{GH\otimes}(B(q)) = \ln \frac{q}{1-q}. \tag{73}$$

For $q > 1$, $U^{(q)}$ is a two-mode squeezing transformation with gain q, which has symplectic matrix (34) and is denoted hereafter by $A(q)$. Then from Corollary 8 we have

$$Q_{GH\otimes}(A(q)) = \max_{\boldsymbol{V}_E} \sup_{\boldsymbol{V}_A} I_c(\boldsymbol{V}_A; A(q)), \tag{74}$$

where the maximization over environment states can again be restricted to states of the form (72). Eqs. (34) and (69) yield $x = q$ and $y = q - 1$ for all one-mode squeezed input environment $\boldsymbol{V}_E$. Invoking (70), we get

$$Q_{GH\otimes}(A(q)) = \ln \frac{q}{q-1}. \tag{75}$$

Both for $q > 1$ and $q < 1$, the formulas recover the infinite capacity of the identity channel in the limit $q \to 1$.

B. Energy-Constrained Passive Environment-Assisted Quantum Capacities

We now move on to energy-constrained quantum capacities. Suppose that P_A (resp. P_E) is the maximum allowed average energy per mode on A system (resp. E system). Then we modify the Definition 2 as follows.

Definition 9: An energy constrained passive environment-assisted quantum code of block length n is a triple $\left(\mathcal{E}^{A_0\to A^n}, \eta^{E^n}, \mathcal{D}^{B^n\to B_0}\right)$ such that, $\operatorname{Tr}\left[\operatorname{Tr}_R \mathcal{E}\left(\Phi^{RA_0}\right)\right] H_{A^n} \le nP_A$ and $\operatorname{Tr}\eta^{(n)} H_{E^n} \le nP_E$.

Its fidelity is given by $\operatorname{Tr}\Phi^{RA_0}\sigma^{RB_0}$ and its rate by the number of modes of A_0 over n.

A rate $\mathtt{R}$ is called achievable if there are codes for all block lengths n with fidelity converging to 1 and rate converging to $\mathtt{R}$. The energy constrained passive environment-assisted quantum capacity of W, denoted by $Q_H(W; P_A; P_E)$ is the supremum of all achievable rates.

If the helper is restricted to fully separable states η^{E^n}, i.e. convex combinations of tensor products $\eta^{E^n} = \eta^{E^1} \otimes \ldots \otimes \eta^{E^n}$, the supremum of all achievable rates is denoted by $Q_{H\otimes}(W; P_A; P_E)$.

Theorem 10: For a Gaussian isometry $W : AE \to BF$, the energy-constrained passive environment-assisted quantum capacity is given by

$$\begin{aligned} Q_H(W; P_A; P_E) &= \sup_n \sup_{\eta^{(n)}} \frac{1}{n} Q\Big(\mathcal{N}_{\eta^{(n)}}^{\otimes n}, nP_A\Big) \\ &= \sup_n \sup_{\eta^{(n)}} \max_{\rho^{(n)}} \frac{1}{n} I_c\Big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big), \end{aligned} \tag{76}$$

where the maximization is over states $\rho^{(n)}$ on A^n with $\operatorname{Tr}\rho^{(n)} H_{A^n} \le nP_A$ and states $\eta^{(n)}$ on E^n with $\operatorname{Tr}\eta^{(n)} H_{E^n} \le nP_E$.

The capacity with separable helper is given by the same formula, but now varying only over product states $\eta^{(n)} = \eta_1 \otimes \ldots \otimes \eta_n$ and respecting the energy constraints $\operatorname{Tr}\rho^{(n)} H_{A^n} \le nP_A$ and $\sum_{i=1}^n \operatorname{Tr}\eta_i H_{E_i} \le nP_E$. Consequently, $Q_H(W; P_A; P_E) = \lim_{n\to\infty} \frac{1}{n} Q_{H\otimes}(W; nP_A; nP_E)$.

Proof: Considering the Hamiltonian operator $H_{A^nE^n} = H_{A^n} \otimes \mathbb{1}_{E^n} + \mathbb{1}_{A^n} \otimes H_{E^n}$ on the system A^nE^n, we have

$$\operatorname{Tr}\rho^{(n)} \otimes \eta^{(n)} H_{A^nE^n} \le nP_A + nP_E, \tag{77}$$

where $\rho^{(n)} \otimes \eta^{(n)}$ is an arbitrary allowed input state to the system A^nE^n. Using the fact that

$$\operatorname{Tr}\exp(-\beta H_{A^n}),\ \operatorname{Tr}\exp(-\beta H_{E^n}) < \infty \text{ for all } \beta > 0, \tag{78}$$

we get

$$\begin{aligned} &\operatorname{Tr}\exp(-\beta H_{A^nE^n}) \\ &= \big(\operatorname{Tr}\exp(-\beta H_{A^n})\big)\big(\operatorname{Tr}\exp(-\beta H_{E^n})\big) \\ &< \infty. \end{aligned} \tag{79}$$

Thus, according to [12], the set $\mathcal{C} = \{\rho^{(n)} \otimes \eta^{(n)} : \operatorname{Tr}\rho^{(n)} \otimes \eta^{(n)} H_{A^nE^n} \le nP_A + nP_E\}$ is compact.

Using [28, Cor. 14] and the fact that

$$\sup_{\rho^{(n)}\otimes\eta^{(n)}\in\mathcal{C}} S(\mathcal{N}^{\otimes n}(\rho^{(n)} \otimes \eta^{(n)})) < \infty, \tag{80}$$

coming from Lemma 1, we see that the coherent information $I_c\Big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big)$, for any fixed $\eta^{(n)}$, is continuous and hence it takes its maximum on the set $\{\rho^{(n)} \mid \operatorname{Tr}\rho^{(n)} H_{A^n} \le nP_A\}$. By applying (80), we then have

$$\begin{aligned} -\infty &< -S(\mathcal{N}_{\eta^{(n)}}^{\otimes n}(\rho^{(n)})) \\ &\le I_c\Big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big) \\ &\le S(\mathcal{N}_{\eta^{(n)}}^{\otimes n}(\rho^{(n)})) \\ &< +\infty. \end{aligned} \tag{81}$$

Therefore, the quantity $Q_H(W; P_A; P_E)$ is finite. ■

Remark 11: If $\eta^{(n)}$ is pure, we have $I_c\Big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big) = I_c\big(\rho^{(n)} \otimes \eta^{(n)}; \mathcal{N}^{\otimes n}\big)$ and the latter is continuous with maximum on $\mathcal{C}$. Consequently, the $\sup_{\eta^{(n)}}$ in Theorem 10 can be turned into $\max_{\eta^{(n)}}$.

Let us evaluate the energy-constrained environment-assisted quantum capacities for unitaries that are universally degradable with respect to Gaussian environment states. To do so, recall

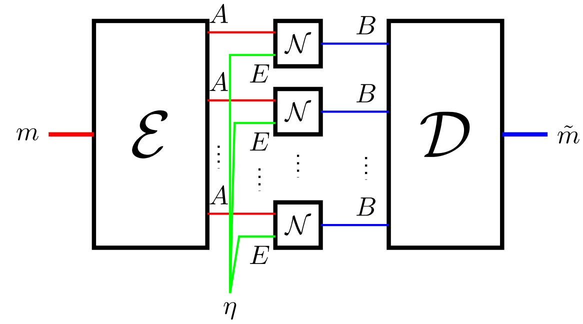

Fig. 3. Schematic of the protocol for transmit classical information with passive assistance from the environment; $\mathcal{E}$ and $\mathcal{D}$ are the encoding and decoding maps respectively.

[33, Thms. 13 and 14] that for a degradable channel $\mathcal{N}^{A\to B}$, the energy-constrained quantum capacity is given by

$$Q(\mathcal{N}, P_A) = \sup_{\rho\,:\,\operatorname{Tr}\rho H_A \le P_A} S\big(\mathcal{N}(\rho)\big) - S\big(\widetilde{\mathcal{N}}(\rho)\big), \tag{82}$$

where the supremum is achieved by the Gibbs state $\gamma_A(P_A)$.

In particular for degradable channels $\mathcal{N}_i$,

$$\begin{aligned} &Q(\mathcal{N}_1 \otimes \ldots \otimes \mathcal{N}_n, nP) \\ &= \max_{\{P_i\}} \sum_i S(\mathcal{N}_i(\gamma_A(P_i))) - S\Big(\widetilde{\mathcal{N}}_i(\gamma_A(P_i))\Big) \\ &\quad \text{s.t.} \sum_i P_i = nP_A, \end{aligned} \tag{83}$$

an optimization that can be performed by Lagrange multipliers in the cases of interest.

For unitaries that are universally degradable with respect to Gaussian environment states, the energy-constrained Gaussian separable environment-assisted capacity is bounded below by

$$\begin{aligned} &Q_{GH\otimes}(U, P_A, P_E) \ge \\ &\max_{\eta_G\,:\,\operatorname{Tr}\eta H_E \le P_E} S(\operatorname{Tr}_F U_{\eta_G}(\gamma_A(P_A))) \\ &\quad - S(\operatorname{Tr}_B U_{\eta_G}(\gamma_A(P_A))). \end{aligned} \tag{84}$$

With this we can find lower bounds for beam splitter and amplifier unitaries, and additionally also find their upper bounds when letting $P_A \to \infty$. For example, the upper bounds for single mode beam splitter and amplifier channels are provided by (73) and (75), respectively.

IV. Classical Communication

In this section, we consider classical communication in the passive environment-assisted model. After deriving the classical capacity, we put forward an uncertainty relation for it, that arises when exchanging the roles of active and passive users. Finally we will briefly discuss conferencing encoders.

Referring to Fig. 3, suppose Alice selects some classical message m from the set of messages $\{1, 2, \ldots, |M|\}$ to communicate to Bob. An encoding CPTP map $\mathcal{E} : M \to \mathcal{T}(A^n)$ can be realized by preparing states $\{\alpha_m\}$ to be input across A^n of n instances of the channel. Here M is a Hilbert space with orthonormal basis $\{|m\rangle\}$. A decoding CPTP map

$\mathcal{D} : \mathcal{T}(B^n) \to M$ can be realized by a positive operator-valued measure (POVM) $\{\Lambda_m\}$. The probability of error for a particular message m, i.e. of decoding $\tilde{m} \neq m$, is

$$P_e(m) = 1 - \mathrm{Tr}\Big[\Lambda_m \mathcal{N}^{\otimes n}\Big(\alpha_m^{A^n} \otimes \eta^{E^n}\Big)\Big]. \quad (85)$$

Definition 12: A passive environment-assisted classical code of block length n is a family of triples $\{\alpha_m^{A^n}, \eta^{E^n}, \Lambda_m\}$ with the error probability $\overline{P}_e := \frac{1}{|M|}\sum_m P_e(m)$ and the rate $\frac{1}{n}\ln|M|$. A rate R is achievable if there is a sequence of codes over their block length n with $\overline{P}_e$ converging to 0 and rate converging to R. The passive environment-assisted classical capacity of W, denoted by $C_H(W)$, is the maximum achievable rate.

If the helper is restricted to fully separable states η^{E^n}, i.e., convex combinations of tensor products $\eta^{E^n} = \eta^{E_1} \otimes \ldots \otimes \eta^{E_n}$, the largest achievable rate is denoted by $C_{H\otimes}(W)$.

Since the error probability is linear in the environment state, without loss of generality the latter may be assumed to be pure, for both unrestricted and separable helper.

Theorem 13: For a Gaussian isometry $W : AE \to BF$, the energy-constrained passive environment-assisted classical capacity is given by

$$C_H(W, P_A, P_E) = \sup_n \max_{\eta^{(n)}} \frac{1}{n} C\Big(\mathcal{N}_{\eta^{(n)}}^{\otimes n}, nP_A\Big), \quad (86)$$

where the maximization is over environment input states $\eta^{(n)}$ respecting energy constraint $\mathrm{Tr}\,\eta^{(n)} H_{E^n} \leq nP_E$.

Similarly, the capacity with separable helper is given by the same formula,

$$C_{H\otimes}(W, P_A, P_E) = \sup_n \max_{\eta^{(n)}=\eta_1\otimes\ldots\otimes\eta_n} \frac{1}{n} C(\mathcal{N}_{\eta_1} \otimes \ldots \otimes \mathcal{N}_{\eta_n}, nP_A), \quad (87)$$

where the maximum is only over product states, i.e. $\eta^{(n)} = \eta_1 \otimes \ldots \otimes \eta_n$ respecting the energy constraint $\mathrm{Tr}\,\eta^{(n)} H_{E^n} \leq nP_E$.

As a consequence of the theorem, we have $C_H(W, P_A, P_E) = \lim_{n\to\infty} \frac{1}{n} C_{H\otimes}(W, nP_A, nP_E)$.

Proof: Consider the Hamiltonian operator $H_{AE} = H_A \otimes \mathbb{1} + \mathbb{1} \otimes H_E$ on the system AE together with

$$\begin{aligned} H_{A^nE^n} := & H_{AE} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes H_{AE} \otimes \cdots \otimes \mathbb{1} \\ & + \ldots + \mathbb{1} \otimes \cdots \otimes H_{AE}. \end{aligned} \quad (88)$$

For a given density matrix $\eta^{(n)} = \eta_1 \otimes \cdots \otimes \eta_n$, we have

$$\begin{aligned} & \sup_{\rho^{(n)}:\mathrm{Tr}\,\rho^{(n)} H_{A^n} \leq nP_A} S\Big(\mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big(\rho^{(n)}\Big)\Big) \\ & \leq \sup_{\rho^{(n)}:\mathrm{Tr}\,\rho^{(n)} H_{A^n} \leq nP_A} S\Big(\mathcal{N}^{\otimes n}\Big(\rho^{(n)} \otimes \eta^{(n)}\Big)\Big) \quad (89) \\ & \leq \sup_{\rho^{(n)}:\mathrm{Tr}\,\rho^{(n)} H_{A^n} \leq nP_A} \sum_{i=1}^n S(\mathcal{N}(\rho_i \otimes \eta_i)) \quad (90) \\ & \leq \sum_{i=1}^n \sup_{\rho_i:\mathrm{Tr}\,\rho_i H_A \leq P_A} S(\mathcal{N}(\rho_i \otimes \eta_i)). \quad (91) \end{aligned}$$

In getting the above sequence of inequalities we exploited the subadditivity of the von Neumann entropy by introducing ρ_i as submarginal of $\rho^{(n)}$, for $i = 1 \ldots n$. Next, we have

$$\begin{aligned} & \sup_{\eta^{(n)}:\mathrm{Tr}(\eta^{(n)} H_{E^n}) \leq nP_E} \; \sup_{\rho^{(n)}:\mathrm{Tr}(\rho^{(n)} H_{A^n}) \leq nP_A} S\Big(\mathcal{N}_{\eta^{(n)}}^{\otimes n}\Big(\rho^{(n)}\Big)\Big) \\ & \leq n^2 \sup_{\eta:\mathrm{Tr}\,\eta H_E \leq P_E} \; \sup_{\rho:\mathrm{Tr}\,\rho H_A \leq P_A} S(\mathcal{N}(\rho \otimes \eta)). \end{aligned} \quad (92)$$

By Lemma 1 the quantity (92) is finite and so is the l.h.s. of (89).

Let us consider $\rho = \int p_x \rho_x dx$ as the average input on a single channel use. Clearly we have

$$\mathrm{Tr}\,\rho H_A \leq \int \mathrm{Tr}\,\rho_x H_A p_x dx \leq P_A. \quad (93)$$

Replacing ρ by $\mathcal{N}_\eta(\rho)$ and using Lemma 1, the Holevo χ-quantity

$$\chi(\{p_x, \mathcal{N}_\eta(\rho_x)\}) = S(\mathcal{N}_\eta(\rho)) - \int S(\mathcal{N}_\eta(\rho_x)) p_x dx, \quad (94)$$

results finite. Then by means of (92), it is clear that

$$C\Big(\mathcal{N}_{\eta^{(n)}}^{\otimes n}, nP_A\Big) = \sup_{p_{x^n},\, \rho_{x^n}^{(n)}} \chi\Big(\Big\{p_{x^n}, \mathcal{N}_\eta^{\otimes n}\Big(\rho_{x^n}^{(n)}\Big)\Big\}\Big), \quad (95)$$

is finite as well and so C_H is correctly defined. Now, the proof of the direct parts, i.e. "$\geq$", follows immediately from the Holevo-Schumacher-Westmoreland theorem [14], [27].

For the converse parts, i.e. "$\leq$", the proof goes like [17, Thm. 1]. ■

For unitaries of most interest, like beam-splitter and amplifier, we can give a lower bound on the classical capacity with separable helper. Let us encode classical stochastic variable m, distributed according to a probability density P_m, into the quantum states ρ_m^A. The modulation due to encoding is given by $\boldsymbol{V}_{mod}$ and $\overline{\boldsymbol{V}}_A = \boldsymbol{V}_A + \boldsymbol{V}_{mod}$ gives the average input state after encoding. We assume that the distribution of the classical messages is a Gaussian distribution with zero mean whose covariance matrix is given by $\boldsymbol{V}_{mod}$. The average energy of the input states in terms of the CM is given by $P_A = \frac{\mathrm{Tr}\,\overline{\boldsymbol{V}}_A}{4n} - \frac{1}{2}$, and likewise $P_E = \frac{\mathrm{Tr}\,\overline{\boldsymbol{V}}_E}{4n} - \frac{1}{2}$ for the environment. Then, for beam splitter and amplifier we get the following form for the environment-assisted capacities when the helper is restricted to separable states in the environment,

$$\begin{aligned} & C_{H\otimes}(U, P_A, P_E) \geq \\ & \max_s \Bigg\{ g\Bigg(|x|P_A + y\cosh(2s) + \frac{|x|-1}{2}\Bigg) \\ & \quad - g\Bigg(y + \frac{|x|-1}{2}\Bigg); P_A \geq P_{\mathrm{th}} \Bigg\}, \end{aligned} \quad (96)$$

where we used the notations x and y from Eq. (69) with $x \neq 0, 1$. Furthermore, $\cosh(2s) \leq 2P_E + 1$ and $P_{\mathrm{th}} = e^{2|s|} + \frac{2y\sinh(2|s|)}{|x|} - 1$. For a general one-mode environment state we can find a symplectic orthogonal transformation, that makes $\boldsymbol{V}_E$ diagonal (this symplectic orthogonal transformation is a rotation, thus the effective state is a squeezed one-mode state), which does not affect the energy constraints on the input environment. Now using [23, Thm. 1], we have $\boldsymbol{V}_A$ and $\boldsymbol{V}_{mod}$ to be diagonal in the same basis as $\boldsymbol{V}_E$. In fact we can choose the seed state of the input to be $\boldsymbol{V}_E$ (in its diagonal form). Then following the calculation in [24], we get the claimed result.

A. Capacities Uncertainty Relation

For a given isometry $W : AE \to BF$, the following quantity corresponds to the *product-state capacity with separable helper*

$$\chi_{H\otimes}(W, P_A, P_E) = \max_{\rho,\eta\,:\,\mathrm{Tr}\,\rho H_A \leq P_A,\,\mathrm{Tr}\,\eta H_E \leq P_E} \chi(\{p_x dx, \mathcal{N}_\eta(\rho_x)\}), \tag{97}$$

where on the r.h.s we have the Holevo χ quantity for the effective channel $\mathcal{N}_\eta^{A\to B}(\rho) := \mathcal{N}^{AE\to B}(\rho \otimes \eta)$ [see Eq. (4)] upon inputting the ensemble $\{p_x dx, \rho_x\}$, and $\rho = \int p_x \rho_x dx$.

Now, besides this channel $A \to B$, we can also define another effective channel $E \to B$ by fixing the state of A and tracing over F, namely $\overline{\mathcal{N}}_\rho^{E\to B}(\eta) := \mathcal{N}^{AE\to B}(\rho \otimes \eta)$ [see again Eq. (4)]. For the latter, the following quantity corresponds to the product-state capacity with separable helper

$$\chi_{A\otimes}(W, P_A, P_E) = \max_{\rho,\eta\,:\,\mathrm{Tr}\,\rho H_A \leq P_A,\,\mathrm{Tr}\,\eta H_E \leq P_E} \chi\big(\{p_x dx, \overline{\mathcal{N}}_\rho(\eta_x)\}\big). \tag{98}$$

Theorem 14: Let $W : A \otimes E \longrightarrow B \otimes F$ be a Gaussian unitary on $N_A + N_E$ modes with associated symplectic matrix of the following canonical form

$$\boldsymbol{S} = \begin{pmatrix} \mathbb{1} & 0 & \mathbb{1} - \boldsymbol{J}^{-\top} & 0 \\ 0 & \boldsymbol{J} & 0 & -\boldsymbol{J} \\ \mathbb{1} & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} - \boldsymbol{J} & 0 & \boldsymbol{J} \end{pmatrix},$$

or

$$\boldsymbol{S} = \begin{pmatrix} \mathbb{1} & 0 & 0 & \boldsymbol{J}^{-\top} - \mathbb{1} \\ 0 & \boldsymbol{J} & -\boldsymbol{J} & 0 \\ 0 & \boldsymbol{J}^{-1} - \mathbb{1} & \mathbb{1} & 0 \\ -\boldsymbol{J}^\top & 0 & 0 & \boldsymbol{J}^\top \end{pmatrix}, \tag{99}$$

where $\mathbb{1}$ and $\boldsymbol{J}$ are $(N_A + N_E) \times (N_A + N_E)$ identity and block-diagonal (in the real Jordan form) matrices respectively. Assuming

$$(\mathbb{1} - \boldsymbol{J}^{-\top})(\mathbb{1} - \boldsymbol{J}^{-1}) \leq \mathbb{1}, \tag{100}$$

and Hamiltonians H_A and H_E for Alice and the helper as in Eq. (13), with an average photon number per mode constrained by P_A and P_E respectively, we have

$$\chi_{A\otimes}(W, P_A, P_E) + \chi_{H\otimes}(W, P_A, P_E) \geq \frac{\min\{P_A, P_E\}}{2\max\{P_E, P_A\} + 1}. \tag{101}$$

Remark 15: The condition (100) is weaker than the degradability condition for canonical forms (99) considered in [5, Eq. 97]. More precisely, the latter is equivalent to the semi-positivity of a block matrix

$$\begin{pmatrix} \boldsymbol{\Xi}_1 & \boldsymbol{\Xi}_2 \\ \boldsymbol{\Xi}_2^\dagger & \boldsymbol{\Xi}_3 \end{pmatrix}, \tag{102}$$

while semi-positivity of block $\boldsymbol{\Xi}_1$ is enough for (100). Thus, a Gaussian channel might be not degradable, while satisfying (101).

Remark 16: Eq. (101) is a kind of uncertainty relation for $\chi_{H\otimes}$ and $\chi_{A\otimes}$, reminiscent of the entropic uncertainty relations for complementary observables (see e.g. [32]), saying that both cannot be arbitrarily small simultaneously.

Proof: Since the involved capacities refer to product states with separable helper, we can consider single systems A, consisting of N_A modes, and E, consisting of N_E modes.

From the relation (44), the covariance matrix of input state for system A changes to the following

$$\boldsymbol{V}_A \mapsto \boldsymbol{V}_B = \boldsymbol{M}\boldsymbol{V}_A\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{V}_E\boldsymbol{N}^\top. \tag{103}$$

Instead, considering as input the system E and as helper A, the corresponding output is obtained by exchanging A and E in the above expression, namely

$$\boldsymbol{V}_E \mapsto \boldsymbol{V}_B = \boldsymbol{M}\boldsymbol{V}_E\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{V}_A\boldsymbol{N}^\top. \tag{104}$$

As input ensembles, we consider coherent states subject to Gaussian distributions with zero mean, whose covariance matrix are given by $\boldsymbol{V}_{A,mod}$ and $\boldsymbol{V}_{E,mod}$ for Alice and the helper, respectively. The action of encoding is described as follows:

$$\begin{aligned} \overline{\boldsymbol{V}}_A &= \boldsymbol{V}_A + \boldsymbol{V}_{A,mod}, \\ \overline{\boldsymbol{V}}_E &= \boldsymbol{V}_E + \boldsymbol{V}_{E,mod}. \end{aligned} \tag{105}$$

The respective average output states are then given by

$$\overline{\boldsymbol{V}}_A \mapsto \overline{\boldsymbol{V}}_B = \boldsymbol{M}\overline{\boldsymbol{V}}_A\boldsymbol{M}^\top + \boldsymbol{N}\overline{\boldsymbol{V}}_E\boldsymbol{N}^\top, \tag{106}$$

$$\overline{\boldsymbol{V}}_E \mapsto \overline{\boldsymbol{V}}_B = \boldsymbol{M}\overline{\boldsymbol{V}}_E\boldsymbol{M}^\top + \boldsymbol{N}\overline{\boldsymbol{V}}_A\boldsymbol{N}^\top. \tag{107}$$

Coherent input states on the systems A and E means $\boldsymbol{V}_A = \boldsymbol{V}_E = \frac{1}{2}\boldsymbol{I}$. Then, using Eqs. (103) and (106), we can find

$$\begin{aligned} \chi_{H\otimes} \geq\; & S\left(\boldsymbol{M}\left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{V}_{A,mod}\right)\boldsymbol{M}^\top + \frac{1}{2}\boldsymbol{N}\boldsymbol{N}^\top\right) \\ & - S\left(\frac{1}{2}\boldsymbol{M}\boldsymbol{M}^\top + \frac{1}{2}\boldsymbol{N}\boldsymbol{N}^\top\right), \end{aligned} \tag{108}$$

and analogously using Eqs. (104) and (107), we can find

$$\begin{aligned} \chi_{A\otimes} \geq\; & S\left(\boldsymbol{N}\left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{V}_{E,mod}\right)\boldsymbol{N}^\top + \frac{1}{2}\boldsymbol{M}\boldsymbol{M}^\top\right) \\ & - S\left(\frac{1}{2}\boldsymbol{M}\boldsymbol{M}^\top + \frac{1}{2}\boldsymbol{N}\boldsymbol{N}^\top\right). \end{aligned} \tag{109}$$

Choosing $\boldsymbol{V}_{A,mod} = P_A\boldsymbol{I}$ for the channel $\mathcal{N}$ and $\boldsymbol{V}_{E,mod} = P_E\boldsymbol{I}$ for the channel $\overline{\mathcal{N}}$, we get

$$\begin{aligned} \chi_{H\otimes} \geq\; & S\left(\left(P_A + \frac{1}{2}\right)\boldsymbol{M}\boldsymbol{M}^\top + \frac{1}{2}\boldsymbol{N}\boldsymbol{N}^\top\right) \\ & - S\left(\frac{1}{2}\boldsymbol{M}\boldsymbol{M}^\top + \frac{1}{2}\boldsymbol{N}\boldsymbol{N}^\top\right), \end{aligned} \tag{110}$$

and

$$\begin{aligned} \chi_{A\otimes} \geq\; & S\left(\left(P_E + \frac{1}{2}\right)\boldsymbol{N}\boldsymbol{N}^\top + \frac{1}{2}\boldsymbol{M}\boldsymbol{M}^\top\right) \\ & - S\left(\frac{1}{2}\boldsymbol{M}\boldsymbol{M}^\top + \frac{1}{2}\boldsymbol{N}\boldsymbol{N}^\top\right). \end{aligned} \tag{111}$$

Now define the functions

$$f(t) := \mathrm{str}\Big(t\boldsymbol{M}\boldsymbol{M}^\top + \frac{1}{2}\boldsymbol{N}\boldsymbol{N}^\top\Big),$$

and

$$h(t) := \mathrm{str}\Big(\frac{1}{2}\boldsymbol{M}\boldsymbol{M}^\top + t\boldsymbol{N}\boldsymbol{N}^\top\Big) = 2tf\Big(\frac{1}{4t}\Big), \tag{112}$$

where str denotes the symplectic trace, i.e. $\mathrm{str}(A) = \sum_i \nu_i(\boldsymbol{A})$, with $\nu_i(\boldsymbol{A})$ the symplectic eigenvalues of $\boldsymbol{A}$. Notice that these functions are strictly increasing with respect to the parameter t, and so they are invertible functions.

By the Cauchy-Lagrange mean value theorem, for the function $g(x)$, we know there exists a $c \in (a, b)$ such that

$$g(b) - g(a) = g'(c)(b - a). \tag{113}$$

Thus, by choosing $t_b = f^{-1}(b)$, $t_a = f^{-1}(a)$ and $c_1 = f^{-1}(c)$, we get

$$g(f(t_b)) - g(f(t_a)) = g'(f(c_1))(f(t_b) - f(t_a)). \tag{114}$$

Consequently we can write

$$\begin{aligned} & g\Big(f\Big(P_A + \frac{1}{2}\Big)\Big) - g\Big(f\Big(\frac{1}{2}\Big)\Big) \\ &= \Big[f\Big(P_A + \frac{1}{2}\Big) - f\Big(\frac{1}{2}\Big)\Big] \ln\Big(\frac{f(c_1) + \frac{1}{2}}{f(c_1) - \frac{1}{2}}\Big) && (115) \\ &\geq \frac{f\big(P_A + \frac{1}{2}\big) - f\big(\frac{1}{2}\big)}{f(c_1)} && (116) \\ &\geq \frac{f\big(P_A + \frac{1}{2}\big) - f\big(\frac{1}{2}\big)}{f\big(P_A + \frac{1}{2}\big)} && (117) \\ &\geq \frac{1}{P_A + \frac{1}{2}} \frac{f\big(P_A + \frac{1}{2}\big) - f\big(\frac{1}{2}\big)}{f\big(\frac{1}{2}\big)}. && (118) \end{aligned}$$

From Eqs. (115) to (116) we used the elementary relation $x \ln \frac{x+\frac{1}{2}}{x-\frac{1}{2}} \geq 1$, valid for $x \geq \frac{1}{2}$. From Eqs. (116) to (118) we used the property $\mathrm{str}(\boldsymbol{A}) \geq \mathrm{str}(\boldsymbol{B})$, valid for symplectic matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{A} \geq \boldsymbol{B}$ [2]. Analogously, by choosing in Eq. (113), $t_b = h^{-1}(b)$, $t_a = h^{-1}(a)$ and $c_2 = h^{-1}(c)$, we get

$$g(h(t_b)) - g(h(t_a)) = g'(h(c_2))(h(t_b) - h(t_a)). \tag{119}$$

As a consequence, we can write

$$g\Big(h\Big(P_E + \frac{1}{2}\Big)\Big) - g\Big(h\Big(\frac{1}{2}\Big)\Big) \geq \frac{1}{P_E + \frac{1}{2}} \frac{h\big(P_E + \frac{1}{2}\big) - h\big(\frac{1}{2}\big)}{h\big(\frac{1}{2}\big)}. \tag{120}$$

Assuming for the moment $P_A \leq P_E$, and taking into account that f and h are increasing functions, together with the fact that $f\big(\frac{1}{2}\big) = h\big(\frac{1}{2}\big)$, we obtain from Eqs. (118) and (120)

$$\begin{aligned} & g\Big(f\Big(P_A + \frac{1}{2}\Big)\Big) - g\Big(f\Big(\frac{1}{2}\Big)\Big) + g\Big(h\Big(P_E + \frac{1}{2}\Big)\Big) \\ & - g\Big(h\Big(\frac{1}{2}\Big)\Big) \\ &\geq \frac{1}{P_E + \frac{1}{2}} \frac{f\big(P_A + \frac{1}{2}\big) + h\big(P_E + \frac{1}{2}\big) - 2f\big(\frac{1}{2}\big)}{f\big(\frac{1}{2}\big)} \\ &\geq \frac{P_A}{P_E + \frac{1}{2}} \frac{\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top\Big) + \mathrm{str}\Big(\boldsymbol{N}\boldsymbol{N}^\top\Big)}{\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{N}^\top\Big)}. && (121) \end{aligned}$$

By means of Eq. (121) we immediately arrive at

$$\chi_{H\otimes} + \chi_{A\otimes} \geq \frac{P_A}{P_E + \frac{1}{2}} \frac{\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top\Big) + \mathrm{str}\Big(\boldsymbol{N}\boldsymbol{N}^\top\Big)}{\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{N}^\top\Big)}. \tag{122}$$

Now, using the assumption (100), we have

$$\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{N}^\top\Big) \leq 2\,\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top\Big). \tag{123}$$

Finally, replacing this in (122), we arrive at

$$\begin{aligned} \chi_{H\otimes} + \chi_{A\otimes} &\geq \frac{P_A}{P_E + \frac{1}{2}} \frac{\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top\Big) + \mathrm{str}\Big(\boldsymbol{N}\boldsymbol{N}^\top\Big)}{2\,\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top\Big)} \\ &\geq \frac{P_A}{2P_E + 1}. && (124) \end{aligned}$$

■

Remark 17: Relaxing the requirement (100), we may have the following:

- Relation (101) still holds if $\boldsymbol{M}$ and $\boldsymbol{N}$ have diagonal form such that

$$\mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top\Big) + \mathrm{str}\Big(\boldsymbol{N}\boldsymbol{N}^\top\Big) = \mathrm{str}\Big(\boldsymbol{M}\boldsymbol{M}^\top + \boldsymbol{M}\boldsymbol{M}^\top\Big). \tag{125}$$

- Relation (101) still holds if $\boldsymbol{M}\boldsymbol{M}^\top \leq \boldsymbol{N}\boldsymbol{N}^\top$ or $\boldsymbol{M}\boldsymbol{M}^\top \geq \boldsymbol{N}\boldsymbol{N}^\top$.
- A bound tighter than Eq. (101) exists if $N_A = N_E = 1$ (see Appendix B).

In conclusion, unless one of the two energy constraints P_A and P_E is zero, the sum of the classical capacities with helper is always strictly greater than zero. On the other hand, if one of P_A or P_B is zero, the identity or the SWAP unitary show that it can happen that both capacities are zero.

B. Conferencing Encoders

Here we consider conferencing encoders, that is a situation where Alice and the helper can freely communicate classical messages, to prepare signal states for the transmission of a common message (see Fig. 4). The classical capacity with

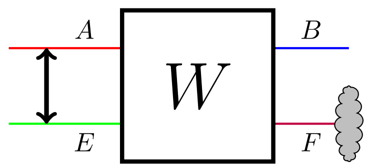

Fig. 4. Diagrammatic view of the parties involved in the communication with conferencing encoders. Differently from Fig. 1, here the party controlling the environment input system E and the sender A can freely communicate classically.

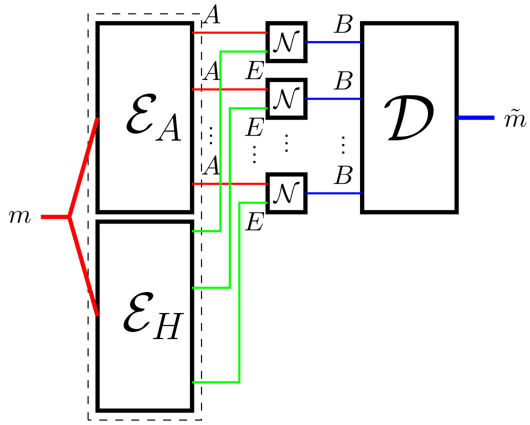

Fig. 5. Schematic of the protocol for transmitting classical information with conferencing encoders; $\mathcal{E}_A$ and $\mathcal{E}_H$ are the encoding maps of Alice and helper respectively. The decoding map is $\mathcal{D}$.

conferencing encoders is then defined in such a way that the encoders (Alice and the helper) are restricted to use product states between A and E.

By referring to Fig. 5, an encoding CPTP map $\mathcal{E} : M \to \mathcal{T}(A^n) \otimes \mathcal{T}(E^n)$ can be thought of as two local encoding maps performed by Alice and Helen, respectively, and given by $\mathcal{E}_A : M \to \mathcal{T}(A^n)$ and $\mathcal{E}_H : M \to \mathcal{T}(E^n)$. These can be realized by preparing pure product states $\{|\alpha_m\rangle \otimes |\eta_m\rangle\}$ to be input across A^n and E^n of n instances of the channel. A decoding CPTP map $\mathcal{D} : \mathcal{T}(B^n) \to M$ can be realized by a POVM $\{\Lambda_m\}$. The probability of error for a particular message m is

$$P_e(m) = 1 - \mathrm{Tr}\Big(\Lambda_m \mathcal{N}^{\otimes n}\Big(\alpha_m^{A^n} \otimes \eta_m^{E^n}\Big)\Big). \tag{126}$$

Definition 18: A classical code for conferencing encoders of block length n is a family of triples $\{|\alpha_m\rangle^{A^n}, |\eta_m\rangle^{E^n}, \Lambda_m\}$ with the error probability $\overline{P}_e := \frac{1}{|M|}\sum_m P_e(m)$ and rate $\frac{1}{n}\ln|M|$. A rate $\mathtt{R}$ is achievable if there is a sequence of codes over their block length n with $\overline{P}_e$ converging to 0 and rate converging to $\mathtt{R}$. The classical capacity with conferencing encoders of W, denoted by $C_{☎}(W)$ is the maximum achievable rate. If the sender and helper are restricted to fully separable states $\alpha_m^{A^n}$ and $\eta_m^{E^n}$, i.e., convex combinations of tensor products $\alpha_m^{A^n} = \alpha_{1m}^{A^1} \otimes \ldots \otimes \alpha_{nm}^{A^n}$ and $\eta_m^{E^n} = \eta_{1m}^{E^1} \otimes \ldots \otimes \eta_{nm}^{E^n}$, for all m, the largest achievable rate is denoted by $C_{☎\otimes}(W)$ and is henceforth referred to as classical capacity with product conferencing encoders.

Theorem 19: For a Gaussian isometry $W : AE \to BF$, satisfying the condition the classical capacity with conferencing encoders is given by

$$C_{☎}(W, P_A, P_E) = \sup_n \max_{\{p(x^n),\, \alpha_{x^n}^{A^n} \otimes \eta_{x^n}^{E^n}\}} \frac{1}{n}\chi\Big(\Big\{p(x^n), \mathcal{N}^{\otimes n}(\alpha_{x^n}^{A^n} \otimes \eta_{x^n}^{E^n})\Big\}\Big), \tag{127}$$

where the maximization is over ensembles respecting energy constraints $\sum_{x^n} p(x^n)\,\mathrm{Tr}(\alpha_{x^n}^{A^n} H_{A^n}) \leq nP_A$ and $\sum_{x^n} p(x^n)\,\mathrm{Tr}(\eta_{x^n}^{E^n} H_{E^n}) \leq nP_E$.

Similarly, the product state capacity of conferencing encoders is given by the formula,

$$C_{☎\otimes}(W, P_A, P_E) = \max_{\{p(x),\, \alpha_x^A \otimes \eta_x^E\}} \chi\big(\{p(x), \mathcal{N}(\alpha_x^A \otimes \eta_x^E)\}\big), \tag{128}$$

where the maximization is over ensembles respecting energy constraints $\sum_x p(x)\,\mathrm{Tr}(\alpha_x^A H_A) \leq P_A$ and $\sum_x p(x)\,\mathrm{Tr}(\eta_x^E H_E) \leq P_E$.

Proof: The direct part, i.e. the "$\geq$" inequality, follows from the HSW Theorem [14], [27]. For the converse part, i.e. the "$\leq$" inequality, the proof goes like that of [17, Thm. 4]. ∎

A lower bound on the classical capacity with conferencing encoders follows from the uncertainty relation of Theorem 14. In fact from the definition of the conferencing encoder, we obtain directly

$$C_{☎\otimes} \geq \max\{\chi_{H\otimes}(W), \chi_{A\otimes}(W)\}, \tag{129}$$

and thus

$$C_{☎\otimes} \geq \frac{\chi_{H\otimes}(W) + \chi_{A\otimes}(W)}{2} \geq \frac{1}{2}\frac{\min\{P_A, P_E\}}{2\max\{P_E, P_A\} + 1}. \tag{130}$$

In other words, the classical capacity with conferencing encoders is always positive, provided the energy is non-zero on both inputs.

Consider a symplectic transformation $\boldsymbol{S}$, given in the block form Eq. (43). Consider seed states with covariance matrices $\boldsymbol{V}_A$ and $\boldsymbol{V}_E$ with zero vector mean. Suppose the classical message is encoded by applying displacement operator to the seed states. We assume that the distribution of the classical messages is a Gaussian distribution with zero mean whose covariance matrix is given by $\boldsymbol{V}_{mod}$. The action of encoding is described as follows:

$$\overline{\boldsymbol{V}}_A = \boldsymbol{V}_A + \boldsymbol{V}_{mod}, \quad \overline{\boldsymbol{V}}_E = \boldsymbol{V}_E + \boldsymbol{V}_{mod}. \tag{131}$$

The covariance matrices of the output state and the output averaged state are labelled $\boldsymbol{V}_B$ and $\overline{\boldsymbol{V}}_B$ respectively and given by

$$\boldsymbol{V}_B = \boldsymbol{M}\boldsymbol{V}_A\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{V}_E\boldsymbol{N}^\top, \quad \overline{\boldsymbol{V}}_B = \boldsymbol{M}\overline{\boldsymbol{V}}_A\boldsymbol{M}^\top + \boldsymbol{N}\overline{\boldsymbol{V}}_E\boldsymbol{N}^\top. \tag{132}$$

Let us evaluate the transmission of classical information by conference encoders using the seed states $\boldsymbol{V}_A = \boldsymbol{V}_E = \boldsymbol{I}/2$ and $\boldsymbol{V}_{mod} = c\boldsymbol{I}/2$.

Imposing the input energy constraint we have (assuming that Alice and the helper are bounded by same energy) in terms of covariance matrices:

$$\frac{\mathrm{Tr}\,\overline{\boldsymbol{V}}_A}{2n} \leq P_A + \frac{1}{2}. \tag{133}$$

Choosing $c = 2\ P_A$ we get the Holevo function of this ensemble to be

$$\sum_{i=1}^{n}\left[g\left(\frac{(2P_A+1)\nu_i - 1}{2}\right) - g\left(\frac{\nu_i - 1}{2}\right)\right], \tag{134}$$

where ν_i are the symplectic eigenvalues of $\boldsymbol{M}\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{N}^\top$. As g is concave monotonic in the argument we have the above quantity non-zero whenever $P_A > 0$. In particular, for the case of beam-splitter, amplifier and conjugate amplifier, $\boldsymbol{M}\boldsymbol{M}^\top + \boldsymbol{N}\boldsymbol{N}^\top = \boldsymbol{I}$, we have the classical information transmission for the above setting given by $g(P_A)$, which is the transmission of ideal channel with mean photon number P_A.

V. Continuity of Capacities in Communication Assisted by Helper

The quantum and classical capacities assisted by separable helper that we defined and studied above also satisfy uniform continuity.

Theorem 20: For input and output energy-limited Gaussian channels $\mathcal{N}^{AE\to B}$ and $\mathcal{M}^{AE\to B}$, if $\|\mathcal{N}^{AE\to B} - \mathcal{M}^{AE\to B}\|_\diamond \leq 2\epsilon$, then

$$\begin{aligned}|C_{H\otimes}(\mathcal{N}) - C_{H\otimes}(\mathcal{M})| &\leq 28\sqrt{\epsilon}\, S\left(\gamma_B\left(\frac{4P_B}{\sqrt{\epsilon}}\right)\right) \\ &\quad + 3g\left(\sqrt{\epsilon} + \frac{1}{2}\right) \qquad (135)\\ |Q_{H\otimes}(\mathcal{N}) - Q_{H\otimes}(\mathcal{M})| &\leq 28\sqrt{\epsilon}\, S\left(\gamma_B\left(\frac{4P_B}{\sqrt{\epsilon}}\right)\right) \\ &\quad + 3g\left(\sqrt{\epsilon} + \frac{1}{2}\right), \qquad (136)\end{aligned}$$

where g is given in Eq. (23) and $\gamma_X(P)$ is the Gibbs state of system X.

Proof: The proof immediately follows from [37, Thm. 9] by noticing that

$$\begin{aligned}\|\mathcal{N}_\eta^{A\to B} - \mathcal{M}_\eta^{A\to B}\|_\diamond &\leq \|\mathcal{N}^{AE\to B} - \mathcal{M}^{AE\to B}\|_\diamond \\ &\leq 2\varepsilon, \quad \forall \eta \in E \end{aligned} \tag{137}$$

the channels $\mathcal{N}_\eta^{A\to B}$, $\mathcal{M}_\eta^{A\to B}$ being restrictions of $\mathcal{N}^{AE\to B}$, $\mathcal{M}^{AE\to B}$ respectively. Furthermore, for any $\eta \in E$, the energy limitation for the output state, according to Eq. (46) of Lemma 1, will be as follows

$$\begin{aligned}\mathrm{Tr}\,\mathcal{N}_\eta(\rho)H_B &= \mathrm{Tr}\,\mathcal{N}^{AE\to B}(\rho\otimes\eta)H_B \\ &\leq 2P_A + 2P_E \equiv P_B,\end{aligned} \tag{138}$$

thus concluding the proof. ■

Remark 21: If we take $\boldsymbol{d}_\rho = \boldsymbol{d}_\eta = 0$ in Lemma 1, then we have $\mathrm{Tr}\,\mathcal{N}_\eta(\rho)H_B \leq P_A + cc_E P_E$. By choosing $c > 0$ such that $cc_E \leq \alpha$ together with $H_B = c\hat{\boldsymbol{r}}_B\hat{\boldsymbol{r}}_B^\top$, the quantity $P_B = P_A + \alpha P_E$ plays the role of $\tilde{E} = \alpha E + E_0$ in [37].

VI. Conclusion

We have created a model of communication via infinite-dimensional channels defined by a bipartite unitary, when assisted by a passive helper in the environment. In this model, we have investigated quantum and classical capacities, proving various general capacity theorems, the former without and with energy constraints, the latter with energy constraints, with respect to natural assumptions on the Hamiltonians involved.

In particular, in Bosonic Gaussian systems, where the Hamiltonian is that of several quantum harmonic oscillators and with a Gaussian unitary defining the interaction, we showed that the capacity formulas lead to simple expressions, when the helper is restricted to Gaussian states. Furthermore, for the classical capacity we showed a tradeoff ("uncertainty") relation between the capacity of Alice assisted by the helper, and that of the helper assisted by Alice in terms of the respective input powers, and a lower bound on the classical capacity with conferencing encoders Alice and helper.

Practically all of our general capacity formulas are multi-letter, and it remains to find bipartite unitaries for which any of them is both non-trivial and explicitly computable, or at least a single-letter formula. In that respect, although we proved the impossibility of having a universally (anti-)degradable Gaussian unitary, it remains open the possibility that for every environment state η the effective channel $\mathcal{N}_\eta$ has a well defined degradability property (not the same for all η). More generally, we would like to know unitaries that are universally degradable (not just for Gaussian helper inputs), for a single-letter quantum capacity, and likewise unitaries resulting in universally additive channels for the Holevo capacity. The lower bound on conferencing encoders based on the capacity uncertainty relation seems very weak, and it remains open to prove better bounds.

Finally, it could be interesting to turn the role of helper into that of an adversary and study how the quantum communication capabilities between Alice and Bob will be hampered by this adversary and the energy constraint it applies. In this sense, the presented model paves the way to investigate arbitrarily varying quantum channels also in infinite dimensional spaces, a topic of particular relevance for the secrecy of practical (in fiber and free space) quantum communication.

Appendix A
Proof of Theorem 7

The proof is divided into two parts, one concerning the case $q < 1$ and another the case $q > 1$. In the former, for $\frac{1}{\sqrt{2}} \leq q < \frac{1}{2} + \frac{\sqrt{3}}{6}$, we obtain a special convex combination of quantum density matrices which has an image through Γ with some negative eigenvalues. Since the method for other cases within the interval $1/2 \leq q < 1$ is similar, we just numerically show the negativity of some eigenvalues for the images through Γ of special convex combinations of density matrices. In contrast, the case $q > 1$ is different, as a contradiction is achieved based on the fact that the quantum relative entropy cannot increase by quantum operations.

A. Case $q < 1$

Proposition 22: The two-mode Gaussian unitaries $U^{(q)}$ for $\frac{1}{\sqrt{2}} \leq q < \frac{1}{2} + \frac{\sqrt{3}}{6}$ are neither universally degradable, nor universally anti-degradable.

Proof: It is enough to prove that there exists a state η_E for which the channel $\mathcal{N}_{\eta_E}^{A\to B}$ is anti-degradable. In view of Remark 4, this state is necessarily non-Gaussian.

The $U^{(q)}$ corresponding to (33) turns out to be

$$U^{(q)} = e^{\arccos\sqrt{q}\left(\hat{a}^\dagger\hat{b}-\hat{a}\hat{b}^\dagger\right)}, \tag{139}$$

for $q \in (0,1)$. Then, for the Fock state $|n\rangle|1\rangle$, we have

$$U^{(q)}|n\rangle|1\rangle = -\frac{1}{\sqrt{(n+1)(1-q)}}\sum_{\ell=0}^{n+1}(-1)^\ell\sqrt{\binom{n+1}{\ell}} (1-q)^{\ell/2}q^{\frac{n-\ell}{2}}((n+1)(1-q)-\ell)|n+1-\ell\rangle|\ell\rangle. \tag{140}$$

By selecting $n = 0, 1$, we get

$$U^{(q)}|0\rangle|1\rangle = -\frac{1}{\sqrt{1-q}}\Big((1-q)|1\rangle|0\rangle + \sqrt{q(1-q)}|0\rangle|1\rangle\Big), \tag{141}$$

and

$$U^{(q)}|1\rangle|1\rangle = -\frac{1}{\sqrt{2(1-q)}}\Big(2\sqrt{q}(1-q)|2\rangle|0\rangle - \sqrt{2}\sqrt{1-q}(1-2q)|1\rangle|1\rangle - 2(1-q)\sqrt{q}|0\rangle|2\rangle\Big). \tag{142}$$

Consider now the channel with environment in the Fock state $|1\rangle\langle 1|$, i.e.

$$\mathcal{N}_q(\rho) = \mathrm{Tr}_E\Big(U^{(q)}(\rho\otimes|1\rangle\langle 1|)U^{(q)\dagger}\Big). \tag{143}$$

Let us assume that there exists a channel Γ such that

$$\Gamma\circ\mathcal{N}(\rho) = \widetilde{\mathcal{N}}(\rho). \tag{144}$$

Inputting $\rho = |0\rangle\langle 0|$, we find that

$$\mathcal{N}_q(|0\rangle\langle 0|) = q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|, \tag{145}$$

and

$$\widetilde{\mathcal{N}}_q(|0\rangle\langle 0|) = q|1\rangle\langle 1| + (1-q)|0\rangle\langle 0|. \tag{146}$$

Therefore, according to (144), we should have

$$q\Gamma(|0\rangle\langle 0|) + (1-q)\Gamma(|1\rangle\langle 1|) = q|1\rangle\langle 1| + (1-q)|0\rangle\langle 0|. \tag{147}$$

Analogously, inputting $\rho = |1\rangle\langle 1|$, we get

$$\mathcal{N}_q(|1\rangle\langle 1|) = \frac{1}{2-2q}\Big(4q(1-q)^2|0\rangle\langle 0| + 2(1-q)(1-2q)^2|1\rangle\langle 1| + 4q(1-q)^2|2\rangle\langle 2|\Big), \tag{148}$$

and

$$\widetilde{\mathcal{N}}_q(|1\rangle\langle 1|) = \frac{1}{2-2q}\Big(4q(1-q)^2|2\rangle\langle 2| + 2(1-q)(1-2q)^2|1\rangle\langle 1| + 4q(1-q)^2|0\rangle\langle 0|\Big). \tag{149}$$

Hence, according to (144), we should have

$$\frac{1}{2-2q}\Big(4q(1-q)^2\Gamma(|0\rangle\langle 0|) + 2(1-q)(1-2q)^2\Gamma(|1\rangle\langle 1|) + 4q(1-q)^2\Gamma(|2\rangle\langle 2|)\Big) = \frac{1}{2-2q}\Big(4q(1-q)^2|2\rangle\langle 2| + 2(1-q)(1-2q)^2|1\rangle\langle 1| + 4q(1-q)^2|0\rangle\langle 0|\Big). \tag{150}$$

Now, from (147), we derive

$$\Gamma(|1\rangle\langle 1|) = \frac{q}{1-q}|1\rangle\langle 1| + |0\rangle\langle 0| - \frac{q}{1-q}\Gamma(|0\rangle\langle 0|), \tag{151}$$

which, inserted into (150), yields

$$q\big(1-2q^2\big)\Gamma(|0\rangle\langle 0|) + 2q(1-q)^2\Gamma(|2\rangle\langle 2|) = 2q(1-q)^2|2\rangle\langle 2| + (1-2q)^3|1\rangle\langle 1| + (1-q)(-1+6q-6q^2)|0\rangle\langle 0|. \tag{152}$$

Isolating the term $\Gamma(|2\rangle\langle 2|)$ at l.h.s., we arrive at

$$\Gamma(|2\rangle\langle 2|) = -\frac{1-2q^2}{2(1-q)^2}\Gamma(|0\rangle\langle 0|) + |2\rangle\langle 2| + \frac{(1-2q)^3}{2q(1-q)^2}|1\rangle\langle 1| + \frac{-1+6q-6q^2}{2q(1-q)}|0\rangle\langle 0|. \tag{153}$$

At this point, taking a convex combination of $\Gamma(|1\rangle\langle 1|)$ and $\Gamma(|2\rangle\langle 2|)$ must give a positive operator, given that Γ is a CPTP map. Consider then

$$\frac{1-q}{q}\Gamma(|1\rangle\langle 1|) + \frac{2(1-q)^2}{2q^2-1}\Gamma(|2\rangle\langle 2|), \tag{154}$$

with $q \geq \frac{1}{\sqrt{2}}$, we get

$$\begin{aligned}&\frac{1-q}{q}\Gamma(|1\rangle\langle 1|) + \frac{2(1-q)^2}{2q^2-1}\Gamma(|2\rangle\langle 2|)\\ =&\frac{2(1-q)^2}{2q^2-1}|2\rangle\langle 2| + \left[1+\frac{(1-2q)^3}{q(2q^2-1)}\right]|1\rangle\langle 1|\\ &+\left[\frac{1-q}{q}+\frac{(1-q)(-1+6q-6q^2)}{q(2q^2-1)}\right]|0\rangle\langle 0|.\end{aligned} \tag{155}$$

Now, if we analyze the coefficients at r.h.s. (which correspond to the eigenvalues of the convex combination of $\Gamma(|1\rangle\langle 1|)$ and $\Gamma(|2\rangle\langle 2|)$) we have

$$\frac{2(1-q)^2}{2q^2-1} \geq 0 \quad \text{for} \quad \frac{1}{\sqrt{2}} \leq q < 1, \tag{156}$$

$$1+\frac{(1-2q)^3}{q(2q^2-1)} < 0 \quad \text{for} \quad \frac{1}{\sqrt{2}} \leq q < \frac{1}{2}+\frac{\sqrt{3}}{6}, \tag{157}$$

$$\frac{1-q}{q}+\frac{(1-q)(-1+6q-6q^2)}{q(2q^2-1)} > 0 \quad \text{for} \quad \frac{1}{\sqrt{2}} \leq q < 1. \tag{158}$$

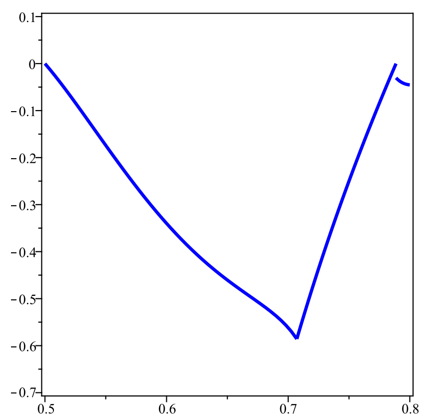

Fig. 6. Quantities $c(k_n, k_m)$ vs q. In particular, in the range $[1/2, 1/\sqrt{2}]$ it is plotted $c(k_4, -k_2)$. In the range $[1/\sqrt{2}, 1/2 + \sqrt{3}/6)$ it is plotted $c(k_2, -k_1)$, according to Proposition 22. Finally, in the range $[1/2 + \sqrt{3}/6, 0.8]$ it is plotted $c(-k_4, k_2)$. In the point $q = 1/2 + \sqrt{3}/6$, it is $c(k_2, -k_1) = 0$ while $c(-k_4, k_2) = -0.0303$.

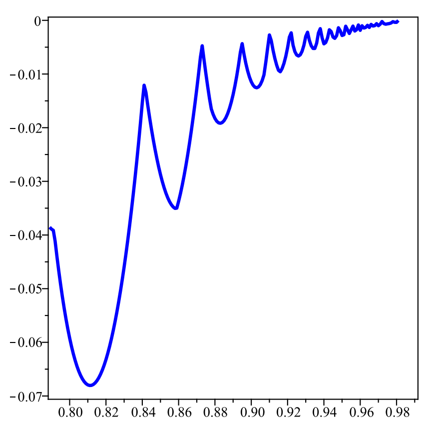

Fig. 7. The quantity $\min_{n,m\le 50} C(|k_n|, |k_m|)$ vs q when $k_n k_m < 0$.

Thus, we can conclude that the channel Γ does not exist (at least for $\frac{1}{\sqrt{2}} \le q < \frac{1}{2} + \frac{\sqrt{3}}{6}$) because its eigenvalues should have been positive. This in turn means that in the above range of q values the Gaussian unitaries are neither universally degradable nor universally anti-degradable. ∎

Remark 23: Numerical investigations (see below) suggests that the statement of Proposition 22 holds actually true for q between $1/2$ and 1.

Let us consider the convex combination of states as

$$\frac{k_m}{k_m + k_n}\Gamma(|n\rangle\langle n|) + \frac{k_n}{k_m + k_n}\Gamma(|m\rangle\langle m|), \tag{159}$$

where k_m and k_n are the coefficients in front of $\Gamma(|0\rangle\langle 0|)$ for the expressions of $\Gamma(|m\rangle\langle m|)$ and $\Gamma(|n\rangle\langle n|)$, respectively.

Define $c(k_n, k_m)$ the coefficient of $|1\rangle\langle 1|$ for the combination (159). Figures 6 and 7 show that there is always a negative $c(k_n, k_m)$ for $q \in [\frac{1}{2}, 1)$.

B. Case $q > 1$

The $U^{(q)}$ corresponding to (34) turns out to be

$$U^{(q)} = e^{i\,\mathrm{arccosh}\sqrt{q}\left(\hat{a}^\dagger\hat{b}^\dagger + \hat{a}\hat{b}\right)}. \tag{160}$$

Using the disentangling formula for the $SU(1,1)$ group, it is possible to rewrite it as

$$U^{(q)} = e^{r\hat{a}^\dagger\hat{b}^\dagger} e^{-s\left(\hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger\right)} e^{r\hat{a}\hat{b}}, \tag{161}$$

where

$$r = i\sqrt{\frac{q-1}{q}}, \quad s = \ln\sqrt{q}. \tag{162}$$

Let us now compute the action of $U^{(q)}$ on the Fock state $|m\rangle|1\rangle$. It results

$$\begin{aligned}
&U^{(q)}|m\rangle|1\rangle \\
&= e^{\hat{a}^\dagger\hat{b}^\dagger r} e^{-\left(\hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger\right)s}\left(\sum_{n=0}^{\infty}\frac{r^n\left(\hat{a}\hat{b}\right)^n}{n!}|m\rangle|1\rangle\right) &(163)\\
&= e^{\hat{a}^\dagger\hat{b}^\dagger r} e^{-\left(\hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger\right)s}\left(|m\rangle|1\rangle + \sqrt{m}r|m-1\rangle|0\rangle\right) &(164)\\
&= e^{\hat{a}^\dagger\hat{b}^\dagger r}\left(\sum_{n=0}^{\infty}\frac{(-1)^n s^n\left(\hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger\right)^n}{n!}\right)\Big(|m\rangle|1\rangle \\
&\quad + \sqrt{m}r|m-1\rangle|0\rangle\Big) &(165)\\
&= e^{\hat{a}^\dagger\hat{b}^\dagger r}\Big(\sum_{n=0}^{\infty}\frac{(-1)^n s^n(m+2)^n}{n!}|m\rangle|1\rangle \\
&\quad + \sqrt{m}r\sum_{n=0}^{\infty}\frac{(-1)^n s^n(m-1+1)^n}{n!}|m-1\rangle|0\rangle\Big) &(166)\\
&= e^{\hat{a}^\dagger\hat{b}^\dagger r}\left(e^{-(m+2)s}|m\rangle|1\rangle + \sqrt{m}re^{-ms}|m-1\rangle|0\rangle\right) &(167)\\
&= e^{-(m+2)s}\sum_{n=0}^{\infty}\sqrt{n+1}\sqrt{\binom{n+m}{m}}r^n|n+m\rangle|n+1\rangle \\
&\quad + \sqrt{m}re^{-ms}\sum_{n=0}^{\infty}\sqrt{\binom{n+m-1}{m-1}}r^n|n+m-1\rangle|n\rangle &(168)\\
&= \sqrt{m}re^{-ms}|m-1\rangle|0\rangle \\
&\quad + \sum_{n=0}^{\infty}\Bigg(e^{-(m+2)s}\sqrt{n+1}\sqrt{\binom{n+m}{m}} \\
&\quad + \sqrt{m}r^2e^{-ms}\sqrt{\binom{n+m}{m-1}}\Bigg)r^n|n+m\rangle|n+1\rangle. &(169)
\end{aligned}$$

Then, we can get

$$\begin{aligned}
&\mathcal{N}(|m\rangle\langle m|) = m|r|^2e^{-2ms}|m-1\rangle\langle m-1| \\
&+ \sum_{n=0}^{\infty}\Bigg(e^{-(m+2)s}\sqrt{n+1}\sqrt{\binom{n+m}{m}} \\
&- \sqrt{m}\frac{q-1}{q}e^{-ms}\sqrt{\binom{n+m}{m-1}}\Bigg)^2|r|^{2n}|n+m\rangle\langle n+m|, &(170)
\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathcal{N}}(|m\rangle\langle m|) &= m|r|^2 e^{-2ms}|0\rangle\langle 0| \\ &+ \sum_{n=0}^{\infty}\left(e^{-(m+2)s}\sqrt{n+1}\sqrt{\binom{n+m}{m}}\right. \\ &\left.-\sqrt{m}\frac{q-1}{q}e^{-ms}\sqrt{\binom{n+m}{m-1}}\right)^2 |r|^{2n}|n+1\rangle\langle n+1|.\end{aligned} \tag{171}$$

It is known that for any completely positive map Γ and two density matrices ρ and σ, the following inequality for quantum relative entropy holds true (contractive property)

$$D(\Gamma(\rho)\|\Gamma(\sigma)) \leq D(\rho\|\sigma). \tag{172}$$

By assuming the degradability condition for $\mathcal{N}$, we should have

$$\begin{aligned}D\Big(\widetilde{\mathcal{N}}(|m_1\rangle\langle m_1|)\Big\|\widetilde{\mathcal{N}}(|m_2\rangle\langle m_2|)\Big) \leq \\ D\big(\mathcal{N}(|m_1\rangle\langle m_1|)\big\|\mathcal{N}(|m_2\rangle\langle m_2|)\big),\end{aligned} \tag{173}$$

for all $m_1 > m_2 \in \mathbb{N}$. From Eqs. (170) and (171), we have

$$\begin{aligned}&D\big(\mathcal{N}(|m_1\rangle\langle m_1|)\big\|\mathcal{N}(|m_2\rangle\langle m_2|)\big) \\ &= m_1|r|^2 e^{-2m_1 s}\ln\frac{m_1|r|^2 e^{-2m_1 s}}{c^q_{m_2(m_1-m_2-1)}} \\ &\quad+ \sum_{n=0}^{\infty} c^q_{m_1 n}\ln\frac{c^q_{m_1 n}}{c^q_{m_2(m_1-m_2+n)}},\end{aligned} \tag{174}$$

and

$$\begin{aligned}&D\Big(\widetilde{\mathcal{N}}(|m_1\rangle\langle m_1|)\Big\|\widetilde{\mathcal{N}}(|m_2\rangle\langle m_2|)\Big) \\ &= m_1|r|^2 e^{-2m_1 s}\ln\frac{m_1|r|^2 e^{-2m_1 s}}{m_2|r|^2 e^{-2m_2 s}} \\ &\quad+ \sum_{n=0}^{\infty} c^q_{m_1 n}\ln\frac{c^q_{m_1 n}}{c^q_{m_2 n}},\end{aligned} \tag{175}$$

where, according to (170) and (171), we have defined

$$\begin{aligned}c^q_{mn} := \left(e^{-(m+2)s}\sqrt{n+1}\sqrt{\binom{n+m}{m}}\right. \\ \left.-\sqrt{m}\frac{q-1}{q}e^{-ms}\sqrt{\binom{n+m}{m-1}}\right)^2 |r|^{2n}.\end{aligned} \tag{176}$$

By simple calculations, we get

$$c^q_{mn} := \frac{(n+1-m(q-1))^2}{(n+1)q^{m+2}}\binom{n+m}{m}\left(\frac{q-1}{q}\right)^n. \tag{177}$$

Then, Eq. (173) reads

$$\begin{aligned}&m_1|r|^2 e^{-2m_1 s}\ln\frac{m_1|r|^2 e^{-2m_1 s}}{m_2|r|^2 e^{-2m_2 s}} + \sum_{n=0}^{\infty} c^q_{m_1 n}\ln\frac{c^q_{m_1 n}}{c^q_{m_2 n}} \\ &\leq m_1|r|^2 e^{-2m_1 s}\ln\frac{m_1|r|^2 e^{-2m_1 s}}{c^q_{m_2(m_1-m_2-1)}} \\ &\quad+ \sum_{n=0}^{\infty} c^q_{m_1 n}\ln\frac{c^q_{m_1 n}}{c^q_{m_2(m_1-m_2+n)}},\end{aligned} \tag{178}$$

or more simply

$$\begin{aligned}m_1|r|^2 e^{-2m_1 s}\ln\frac{m_2|r|^2 e^{-2m_2 s}}{c^q_{m_2(m_1-m_2-1)}} \\ + \sum_{n=0}^{\infty} c^q_{m_1 n}\ln\frac{c^q_{m_2 n}}{c^q_{m_2(m_1-m_2+n)}} \geq 0.\end{aligned} \tag{179}$$

However this inequality can be violated. In fact, it happens that $c^q_{m_2\ n} = 0$ when

$$n = m(q-1) - 1. \tag{180}$$

It is then clear that (179) may be violated when q is close to integer numbers. More precisely the following result holds true.

Theorem 24: For an arbitrary $q > 1$, there exist integers $m_1 > m_2$ such that Eq. (179) is not true.

Proof: Let us consider a fixed rational number $q = \frac{x}{y} > 1$. By selecting $m_2 = y$ and $n' = x - y - 1$, we have

$$n' = m_2(q-1) - 1, \tag{181}$$

that, from (177), guarantees $c^q_{m_2\ n'} = 0$. On the other hand, if there exists n'' such that $c^q_{m_2(n''+m_1-m_2)} = 0$ for such q, then we should have

$$\begin{aligned}\frac{n''+m_1-m_2+1}{m_2} &= q-1 \quad\Rightarrow \\ \frac{n''+m_1-m_2+1}{y} &= \frac{x-y}{y}.\end{aligned} \tag{182}$$

We choose m_1 such that $m_1 - y \neq x - y - n'' - 1$ for any integer $n'' = 0, 1, \ldots$, in order to have $c^q_{m_2(n''+m_1-m_2)} = 0$. We also have $c^q_{m_1\ n'} \neq 0$ to ensure that

$$c^q_{m_1 n'}\ln\frac{c^q_{m_2 n'}}{c^q_{m_2(m_1-m_2+n')}} = -\infty, \tag{183}$$

By considering the above descriptions, we show that relation (179) violates for given small radius $\varepsilon > 0$ and $q < q' < q+\varepsilon$. In other words, it is clear that $c^{q'}_{m_2\ n} \neq 0, \forall\, n$[1] and so we have the condition $\mathrm{supp}(\mathcal{N}(|m_1\rangle\langle m_1|)) \subseteq \mathrm{supp}(\mathcal{N}(|m_2\rangle\langle m_2|))$. Now, for each $n \geq m_2(q-1)+m_1$, we have

$$\begin{aligned}&\frac{c^{q'}_{m_2 n}}{c^{q'}_{m_2(m_1-m_2+n)}} \\ &= \frac{\frac{(n+1-m_2(q'-1))^2}{(n+1)q'^{m_2+2}}}{\frac{(n+m_1-m_2+1-m_2(q'-1))^2}{(n+m_1-m_2+1)q'^{m_2+2}}} \\ &\quad\times\frac{\binom{n+m_2}{m_2}\left(\frac{q'-1}{q'}\right)^n}{\binom{n+m_1-m_2+m_2}{m_2}\left(\frac{q'-1}{q'}\right)^{n+m_1-m_2}}\end{aligned} \tag{186}$$

[1]This holds true for q' irrational number. If q' is a rational number such that $c^q_{m_2\ n} = 0$, we should have

$$\frac{k+1}{m_2} = q' - 1. \tag{184}$$

On the other hand, we have $|q - q'| \leq \varepsilon$ and hence

$$|q-q'| \leq \varepsilon \Rightarrow \left|\frac{x}{m_2} - \frac{k+1+m_2}{m_2}\right| \leq \varepsilon, \tag{185}$$

which implies that $x = k + m_2 + 1$ and so $q' = q$.

$$\leq \frac{\frac{(n+1-m_2(q'-1))^2}{(n+1)}\binom{n+m_2}{m_2}}{\frac{(n+m_1-m_2+1-m_2(q'-1))^2}{(n+m_1-m_2+1)}\binom{n+m_1}{m_2}\left(\frac{q'-1}{q'}\right)^{m_1-m_2}} \tag{187}$$

$$\leq \frac{c^q_{m_2 n}}{c^q_{m_2(m_1-m_2+n)}}\left(\frac{\frac{q-1}{q}}{\frac{q'-1}{q'}}\right)^{m_1-m_2} \tag{188}$$

$$\leq \frac{c^q_{m_2 n}}{c^q_{m_2(m_1-m_2+n)}}. \tag{189}$$

Eq.(188) derives from

$$\frac{n+1-m_2(q'-1)}{n+m_1-m_2+1-m_2(q'-1)} \leq \frac{n+1-m_2(q-1)}{n+m_1-m_2+1-m_2(q-1)}, \tag{190}$$

taking into account that $n \geq m_2(q-1)+m_1$.

On the other hand, we have

$$\lim_{n\to\infty} \frac{c^q_{m_2 n}}{c^q_{m_2(m_1-m_2+n)}} = \frac{1}{|r|^{2m_1}} = \left(\frac{q}{q-1}\right)^{m_1-m_2}. \tag{191}$$

Therefore, for a given $\eta > 0$ the exists a number N_η such that for any $n \geq N_\eta \geq m_2(q-1)+m_1$, it is

$$\frac{c^q_{m_2 n}}{c^q_{m_2(m_1-m_2+n)}} \leq \left(\frac{q}{q-1}\right)^{m_1-m_2} + \eta. \tag{192}$$

It then follows, using (179) and the fact $\mathrm{Tr}(\mathcal{N}(|m_1\rangle\langle m_1|)) = 1$, that

$$\begin{aligned}\sum_{n=N_\eta}^{\infty} c^q_{m_1 n} \ln \frac{c^q_{m_2 n}}{c^q_{m_2(m_1-m_2+n)}} &\leq \ln\left\{\left(\frac{q}{q-1}\right)^{m_1-m_2} + \eta\right\}\sum_{n=0}^{\infty} c_{m_1 n} \\ &\leq \ln\left\{\left(\frac{q}{q-1}\right)^{m_1-m_2} + \eta\right\}.\end{aligned} \tag{193}$$

Using relations (189) and (193), we can get

$$\sum_{n=N_\eta}^{\infty} c^{q'}_{m_1 n} \ln \frac{c^{q'}_{m_2 n}}{c^{q'}_{m_2(m_1-m_2+n)}}$$

$$\leq \sum_{n=N_\eta}^{\infty} c^{q'}_{m_1 n} \ln \frac{c^q_{m_2 n}}{c^q_{m_2(m_1-m_2+n)}} \tag{194}$$

$$\leq \sum_{n=N_\eta}^{\infty} c^{q'}_{m_1 n} \ln \frac{c^q_{m_2 n}}{c^q_{m_2(m_1-m_2+n)}} \tag{195}$$

$$\leq \ln\left\{\left(\frac{q}{q-1}\right)^{m_1-m_2} + \eta\right\}\sum_{n=N_\eta}^{\infty} c^{q'}_{m_1 n} \tag{196}$$

$$\leq \ln\left\{\left(\frac{q}{q-1}\right)^{m_1-m_2} + \eta\right\}. \tag{197}$$

Finally, we find that Eq. (179) holds for $q' \leq \Theta_1 + \Theta_2$, where

$$\begin{aligned}\Theta_1 \equiv\ & m_1|r|^2 e^{-2m_1 s} \ln \frac{m_2|r|^2 e^{-2m_2 s}}{c^{q'}_{m_2(m_1-m_2-1)}} \\ &+ \sum_{n=0, n\neq n'}^{N_\eta - 1} c^{q'}_{m_1 n} \ln \frac{c^{q'}_{m_2 n}}{c^{q'}_{m_2(m_1-m_2+n)}} \\ &+ \ln\left\{\left(\frac{q}{q-1}\right)^{m_1-m_2} + \eta\right\},\end{aligned} \tag{198}$$

$$\Theta_2 \equiv c^{q'}_{m_1 n'} \ln \frac{c^{q'}_{m_2 n'}}{c^{q'}_{m_2(m_1-m_2+n')}}. \tag{199}$$

Now, when ε goes to 0, the quantity Θ_1 will remain finite (it is continuous with respect to q), while the quantity Θ_2 diverges to $-\infty$. Therefore, for any rational number q we can find a set $(q, q+\epsilon)$, for q', which violates (179). Since the set of rational numbers is dense into the set of reals, the proof follows. ■

Appendix B
Bounds on Capacities Uncertainty

In this appendix, we derive, on the basis of relations (110) and (111), tighter lower bounds on the sum $\chi_{H\otimes} + \chi_{A\otimes}$ than the one from Theorem 14 for one-mode Gaussian channels. The reasoning is based on the classification of OMG channels given in [13], and the bounds are derived by using coherent states encoding.

- **Class A1:** $\boldsymbol{M} = 0$, $\boldsymbol{N}\boldsymbol{N}^\top = \boldsymbol{I}$.
  We have

$$\begin{aligned}\chi_{H\otimes} &\geq S\left(\frac{1}{2}\boldsymbol{I}\right) - S\left(\frac{1}{2}\boldsymbol{I}\right) \\ &= 0,\end{aligned} \tag{200}$$

  and

$$\begin{aligned}\chi_{A\otimes} &\geq S\left(\left(P_E + \frac{1}{2}\right)\boldsymbol{I}\right) - S\left(\frac{1}{2}\boldsymbol{I}\right) \\ &\geq g\left(P_E + \frac{1}{2}\right).\end{aligned} \tag{201}$$

  Therefore, we get

$$\chi_{H\otimes} + \chi_{A\otimes} \geq g\left(P_E + \frac{1}{2}\right). \tag{202}$$

- **Class A2:** $\boldsymbol{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\boldsymbol{N}\boldsymbol{N}^\top = \boldsymbol{I}$.
  We have

$$\begin{aligned}\chi_{H\otimes} \geq\ & S\left(\begin{pmatrix} P_A+1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right) \\ &- S\left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right),\end{aligned} \tag{203}$$

  and

$$\begin{aligned}\chi_{A\otimes} \geq\ & S\left(\begin{pmatrix} P_E+1 & 0 \\ 0 & P_E+\frac{1}{2} \end{pmatrix}\right) \\ &- S\left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right).\end{aligned} \tag{204}$$

Therefore, we get

$$\chi_{H\otimes}+\chi_{A\otimes}\geq g\left(\sqrt{(P_A+1)\frac{1}{2}}\right)+g\left(\sqrt{(P_E+1)\left(P_E+\frac{1}{2}\right)}\right)-2g\left(\sqrt{\frac{1}{2}}\right). \tag{205}$$

- **Class B1:** $\boldsymbol{M}=\boldsymbol{I}$, $\boldsymbol{N}\boldsymbol{N}^\top=\frac{1}{2N_0+1}\begin{pmatrix}1&0\\0&0\end{pmatrix}$.
We have

$$\chi_{H\otimes}\geq S\left(\begin{pmatrix}P_A+\frac{1}{2}+\frac{1}{2N_0+1}&0\\0&P_A+\frac{1}{2}\end{pmatrix}\right)-S\left(\begin{pmatrix}\frac{1}{2}+\frac{1}{4N_0+2}&0\\0&\frac{1}{2}\end{pmatrix}\right), \tag{206}$$

and

$$\chi_{A\otimes}\geq S\left(\begin{pmatrix}\frac{P_E+\frac{1}{2}}{2N_0+1}+\frac{1}{2}&0\\0&\frac{1}{2}\end{pmatrix}\right)-S\left(\begin{pmatrix}\frac{1}{2}+\frac{1}{4N_0+2}&0\\0&\frac{1}{2}\end{pmatrix}\right). \tag{207}$$

Therefore, we get

$$\chi_{H\otimes}+\chi_{A\otimes}\geq g\left(\sqrt{\left(P_A+\frac{1}{2}+\frac{1}{2N_0+1}\right)\left(P_A+\frac{1}{2}\right)}\right)+g\left(\sqrt{\frac{P_E+\frac{1}{2}}{4N_0+2}+\frac{1}{2}}\right)-2g\left(\sqrt{\frac{1}{4}+\frac{1}{4N_0+2}}\right). \tag{208}$$

- **Class B2:** $\boldsymbol{M}=\boldsymbol{I}$, $\boldsymbol{N}\boldsymbol{N}^\top=\frac{N_0}{N_0+\frac{1}{2}}\boldsymbol{I}$.
We have

$$\chi_{H\otimes}\geq S\left(\left(P_A+\frac{1}{2}+\frac{N_0}{2N_0+1}\right)\boldsymbol{I}\right)-S\left(\left(\frac{1}{2}+\frac{N_0}{2N_0+1}\right)\boldsymbol{I}\right), \tag{209}$$

and

$$\chi_{A\otimes}\geq S\left(\left(\frac{(P_E+\frac{1}{2})N_0}{N_0+\frac{1}{2}}+\frac{1}{2}\right)\boldsymbol{I}\right)-S\left(\left(\frac{1}{2}+\frac{N_0}{2N_0+1}\right)\boldsymbol{I}\right). \tag{210}$$

Therefore, we get

$$\chi_{H\otimes}+\chi_{A\otimes}\geq g\left(P_A+\frac{1}{2}+\frac{N_0}{2N_0+1}\right)+g\left(\frac{(P_E+\frac{1}{2})N_0}{N_0+\frac{1}{2}}+\frac{1}{2}\right)-2g\left(\frac{1}{2}+\frac{N_0}{2N_0+1}\right). \tag{211}$$

- **Class C Att:** $\boldsymbol{M}=\sqrt{\kappa}\boldsymbol{I}$, $\boldsymbol{N}^\top\boldsymbol{N}=(1-\kappa)\boldsymbol{I}$, $0<\kappa<1$
We have

$$\chi_{H\otimes}\geq S\left(\left(\left(P_A+\frac{1}{2}\right)\kappa+1-\kappa\right)\boldsymbol{I}\right)-S\left(\frac{1}{2}\boldsymbol{I}\right), \tag{212}$$

and

$$\chi_{A\otimes}\geq S\left(\left(\left(P_E+\frac{1}{2}\right)(1-\kappa)+\kappa\right)\boldsymbol{I}\right)-S\left(\frac{1}{2}\boldsymbol{I}\right). \tag{213}$$

Therefore, we get

$$\chi_{H\otimes}+\chi_{A\otimes}\geq g\left(\left(P_A+\frac{1}{2}\right)\kappa+1-\kappa\right)+g\left(\left(P_E+\frac{1}{2}\right)(1-\kappa)+\kappa\right). \tag{214}$$

- **Class C Amp:** $\boldsymbol{M}=\sqrt{\kappa}\boldsymbol{I}$, $\boldsymbol{N}\boldsymbol{N}^\top=(\kappa-1)\boldsymbol{I}$, for $\kappa>1$.
We have

$$\chi_{H\otimes}\geq S\left(\left(\left(P_A+\frac{1}{2}\right)\kappa+\kappa-1\right)\boldsymbol{I}\right)-S\left(\left(\kappa-\frac{1}{2}\right)\boldsymbol{I}\right), \tag{215}$$

and

$$\chi_{A\otimes}\geq S\left(\left(\left(P_E+\frac{1}{2}\right)(\kappa-1)+\kappa\right)\boldsymbol{I}\right)-S\left(\left(\kappa-\frac{1}{2}\right)\boldsymbol{I}\right). \tag{216}$$

Therefore, we get

$$\chi_{H\otimes}+\chi_{A\otimes}\geq g\left(\left(P_A+\frac{1}{2}\right)\kappa+\kappa-1\right)+g\left(\left(P_E+\frac{1}{2}\right)(\kappa-1)+\kappa\right)-2g\left(\kappa-\frac{1}{2}\right). \tag{217}$$

- **Class D:** $\boldsymbol{M}=\sqrt{-\kappa}\boldsymbol{Z}$, $\boldsymbol{N}^\top\boldsymbol{N}=(1-\kappa)\boldsymbol{I}$, $\kappa\in(-\infty,0)$
We have

$$\chi_{H\otimes}\geq S\left(\left(\left(P_A+\frac{1}{2}\right)|\kappa|+|1-\kappa|\right)\boldsymbol{I}\right)-S\left(\frac{|\kappa|+|1-\kappa|}{2}\boldsymbol{I}\right). \tag{218}$$

and

$$\chi_{A\otimes}\geq S\left(\left(\left(P_E+\frac{1}{2}\right)(|1-\kappa|)+|\kappa|\right)\boldsymbol{I}\right)-S\left(\frac{|\kappa|+|1-\kappa|}{2}\boldsymbol{I}\right). \tag{219}$$

Therefore, we obtain

$$\chi_{H\otimes} + \chi_{A\otimes} \geq g\left(\left(P_A + \frac{1}{2}\right)|\kappa| + 1 - \kappa\right) + g\left(\left(P_E + \frac{1}{2}\right)(1-\kappa) + |\kappa|\right) - 2g\left(\frac{|\kappa| + |1-\kappa|}{2}\right). \tag{220}$$

Remark 25: For an easy comparison with the bound in Theorem 14, let us consider the class C. The r.h.s. of (214) and (217) can be put together as

$$g\left(\left(P_A + \frac{1}{2}\right)\kappa + |1-\kappa|\right) + g\left(\left(P_E + \frac{1}{2}\right)|1-\kappa| + \kappa\right) - 2g\left(\frac{|1-\kappa| + \kappa}{2}\right). \tag{221}$$

Due to the properties of the function g defined in (23), it is

$$\text{Eq.(221)} \geq g\left(\left(\min\{P_A, P_E\} + \frac{1}{2}\right)\kappa + |1-\kappa|\right) + g\left(\left(\min\{P_A, P_E\} + \frac{1}{2}\right)|1-\kappa| + \kappa\right) - 2g\left(\frac{|1-\kappa| + \kappa}{2}\right). \tag{222}$$

Still referring to the properties of the function g, we have that the quantity (222) grows, in terms of $\min\{P_A, P_E\}$, faster than (101). Thus, the minimum difference between the two bounds ((221) and (101)) is achieved when $\min\{P_A, P_E\}$ goes to zero and results greater than or equal to

$$\begin{cases} g\left(1 - \frac{1}{2}\kappa\right) + g\left(\frac{1+\kappa}{2}\right), & \kappa < 1, \\ 2g\left(\frac{3}{2}\kappa - 1\right) - 2g\left(\kappa - \frac{1}{2}\right), & \kappa > 1. \end{cases} \tag{223}$$

These two quantities being positive show the tightness of (221) with respect to (101).

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