Generation of Accessible Sets in the Dynamical Modeling of Quantum Network Systems

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***Abstract*—In this article, we consider the dynamical modeling of a class of quantum network systems consisting of qubits, where information extraction is allowed by performing measurement on several selected qubits of the system. For a variety of applications, a state space model is a useful approach to modeling the system dynamics. To construct a state space model for a quantum network system, the major task is to find an accessible set containing all of the operators coupled to the measurement operators. This article focuses on the generation of a proper accessible set for a given system and measurement scheme. We provide analytic results on simplifying the process of generating accessible sets for systems with a time-independent Hamiltonian. Since the order of elements in the accessible set determines the form of state space matrices, guidance is provided to effectively arrange the ordering of elements in the state vector. Defining a system state according to the accessible set, one can develop a state space model with a special pattern inherited from the system structure. As a demonstration, we specifically consider a typical 1-D-chain system with several common measurements and employ the proposed method to determine its accessible set.**

***Index Terms*—Accessible set, dynamical modeling, quantum network system, quantum system.**

I. Introduction

THE dynamical modeling of quantum systems is a basic task for a variety of quantum engineering problems such as quantum identification [1]–[6], [8]–[13], [15]–[18] and quantum control [19]–[21], [24], [26]–[28]. A good dynamical model can benefit the analysis of these problems. This article studies the modeling of a class of quantum network systems whose element systems are qubits and the structure of the system Hamiltonians is given [29], [31], [32]. While the ultimate objective is to generate a state space model for a quantum network system subject to a measurement scheme, our main concern here is on the generation of an efficiently represented accessible set which can be used to generate a state vector. A series of works on Hamiltonian identification, a key task in various applications such as quantum control and quantum computation, has benefited from the state space model commonly used by engineers [9]–[11]. In these results, state space equations were employed to describe the dynamics of a quantum network system and the state space model facilitated developing algorithms to identify unknown parameters in the system Hamiltonian. To formulate the state space equations for the system, a general way is to define a coherence vector [9], [33], which is a complete representation of the quantum state. However, rather than using a complete basis of the operator space, it is often possible to generate an accessible set containing a smaller number of operators necessary for describing the evolution of the measurement operators [9], [34]. Then the state can be defined as a vector of expectation values of operators in the accessible set. A properly generated accessible set may contain the least number of operators that are coupled with the measurement and, thus, significantly reduces the dimension of the system dynamic model. Moreover, a good ordering of element operators in the accessible set can lead to a state space with desired properties which can benefit subsequent analysis. Thus, the generation of accessible sets is important to the system dynamic modeling. Once an accessible set is obtained, the state space equations can then be deduced. Although other methods may also be used to generate a state space model, this article aims to provide a guidance on the generation of accessible sets for obtaining a relatively low-dimensional state space with low computational complexity, which will be instrumental for the efficient modeling of a class of quantum network systems composed of qubit nodes.

The generation of accessible sets is usually complicated. For most cases, the number of elements in an accessible set increases rapidly with the number of subsystems in a network system (see Fig. 1) [29], and, thus, it may be difficult to search for numerical solutions in high-dimensional systems. Although one can always turn to a computer for solutions, the computational complexity can be high. Moreover, the ordering of the elements in the state

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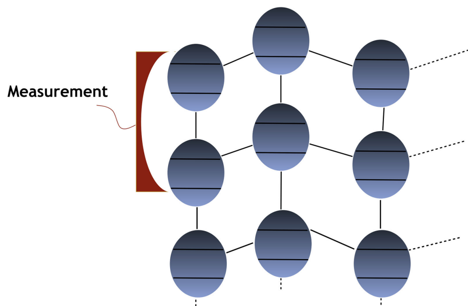

Fig. 1. Example of a quantum network system. The nodes are qubits and each connecting line indicates coupling between two qubits. A measurement device is employed to measure two nodes at the verge of the system.

vector is also nontrivial. Arranging a good ordering of elements in the system state variable may lead to state space matrices with a good structure. In the conference paper [35], preliminary results have been presented in searching for a rapid method for the generation of accessible sets. This article aims at presenting a comprehensive investigation on obtaining good accessible sets while simplifying the generation process. The specific definition of "good" is to give a state space matrix that is easy to analyze and has a repetition pattern as the qubit number increases.

The main contributions of this article are summarized as follows. We first generalize the generation rules of the accessible set to achieve a lower computational complexity. Then we provide several lemmas and propositions to further reduce the computational complexity for a class of spin chain systems. We introduce a powerful tool, graphs, to describe the generation of accessible sets. We prove that the generation of accessible sets can be decomposed as the generation of a series of subsets for a class of quantum chain systems. The division of graphs can help in revealing the repetition pattern of the state space matrices. Graphs can also provide a guidance for the ordering of elements in the state vector. Given the corresponding accessible set, a state space model for the quantum network system can be directly obtained.

The structure of this article is as follows. Section II formulates the problem. Section III presents our main results. A series of illustrative examples are given in Section IV. Section V concludes this article.

II. Preliminaries and Problem Formulation

This work is motivated by establishing simplified state space model for quantum system identification problems (e.g., Hamiltonian identification). In this section, we first provide a brief introduction to Hamiltonian identification and then discuss state space modeling and accessible sets. Finally, problem formulation is presented.

A. Hamiltonian Identification

Hamiltonian is a critical physical quantity determining the evolution of a quantum system. When the system Hamiltonian is unknown or partially known, it can be reconstructed using the Hamiltonian identification technique based on the extracted information (e.g., measurement results) of the system. Lots of efforts have been devoted to developing this technique, and researchers have explored identification algorithms for various quantum systems under different conditions [1]–[6], [8]–[12], [15]–[18].

In the most general case, the Hamiltonian identification problem for quantum network systems is difficult due to the curse of dimensionality, but it can be simplified in structured systems, and a key to this simplification lies in appropriate model establishment. The approaches of state space equation and transfer function have been demonstrated to be two useful methods for modeling and solving Hamiltonian identification problems of quantum network systems [9]–[11]. For example, [9] first provided a rule to generate an accessible set and then derived the corresponding state space model where the system matrix contains the unknown parameters. Then, the input and output information was used to construct a minimal realization of the system by using the eigen-state realization algorithm. Since one system may have nonunique realizations but a unique transfer function, both the modeled and the data-based state space models were further converted into transfer functions. By equating the corresponding coefficients of the two transfer functions, polynomials of the Hamiltonian parameters can be obtained and their solution gives the estimation of the unknown parameters. The method based on the state space model for Hamiltonian identification has two main merits: no state tomography is needed; the algorithm can make use of prior knowledge on the network structure or partial knowledge of the parameters.

Subsequent studies [11], [17] considered the identifiability problem based on the state space model established in [9]. While [11] provided an algorithm based on the transfer function which can be obtained given the state space equations, authors in [17] considered the state space model directly. All these results demonstrated that a good state space model with properly generated coefficient matrices can simplify the subsequent analysis for Hamiltonian identification.

Our research here is motivated by the Hamiltonian identification problem of network systems, aiming to generate a state space model while trying to reduce the model dimension and to reform the coefficient matrices from which clear structured properties can be identified so that the subsequent tasks (e.g., system identification) can be simplified. In this study, we concentrate on the establishment of state space model, where it is critical to generate a proper accessible set. Such a proper state space model may have the potential to facilitate the analysis of a variety of engineering problems including but not limited to Hamiltonian identification and quantum control.

B. State Space Equations and Accessible Sets

Measurement is often needed to extract information about a quantum network system. However, limited by experimental devices, it is common that only part of the network system can be measured in many practical applications (see Fig. 1). For example, one can measure one or two nodes at one edge of the network system to infer information about the whole system.

Given H as the time-independent system Hamiltonian, the time evolution of an arbitrary system observable $O(t)$ in the Heisenberg picture is

$$O(t) = U^{\dagger}(t) O U(t) \tag{1}$$

where $U(t)$ is the unitary operator with time evolution

$$U(t) = e^{-\mathrm{i}Ht}. \tag{2}$$

Here, i is the imaginary unit and t is the time variable (we have set $\hbar = 1$). Taking derivative of both sides of (1), we have

$$\frac{dO(t)}{dt} = \mathrm{i}[H, O(t)]. \tag{3}$$

Given a measurement operator $M(0) = M$, its time evolution is

$$M(t) = e^{\mathrm{i}Ht} M e^{-\mathrm{i}Ht}. \tag{4}$$

According to the Baker–Hausdorff Lemma [36], the Taylor series of $M(t)$ is

$$\begin{aligned} M(t) = M + [H, M]\mathrm{i}t + [H, [H, M]] \frac{(\mathrm{i}t)^2}{2!} \\ + [H, [H, [H, M]]] \frac{(\mathrm{i}t)^3}{3!} + \cdots . \end{aligned} \tag{5}$$

According to (5), the nth-order $(n = 0, 1, 2, \ldots)$ time derivatives of the measurement operator $M(t)$ correspond to

$$M,\ \mathrm{i}[H, M],\ -[H, [H, M]],\ -\mathrm{i}[H, [H, [H, M]]],\ \ldots . \tag{6}$$

To establish a state space model, we aim to find an operator set G constructing a basis for the space spanned by all of the terms in (6). The set G is called the *accessible set* corresponding to the measurement M since all of the element operators in G are accessible by the measurement M. In other words, G is a set of operators whose dynamics are coupled with M. A set of rules to generate the accessible sets is given in [9].

Remark 1: In this article, we assume that a successive record of expectations of the measurement observable M at times $\Delta t, 2\Delta t, 3\Delta, \ldots$ can be obtained, where the sampling period Δt satisfies the Nyquist sampling condition. Such a successive record is referred to as the measurement time trace [9], [13]. In [9], the authors suggested a feasible way to obtain the measurement time trace. Suppose the object quantum system starts its evolution at time zero with state ρ_0. One can apply the measurement operator M at time Δt yielding measurement data y_1. We repeat the procedure of reinitializing the system to ρ_0 and measuring using M after time Δt for n times. When n is large enough, the averaged measurement

$$\frac{1}{n} \sum_{i=1}^{n} y_i$$

is approximately the expectation of the operator M at time Δt. Then, we repeat the whole process with measurement time $2\Delta t$ and the expectation of M at time $2\Delta t$ can be obtained. An experiment for measuring time traces on spin systems has been demonstrated in [13].

Suppose the accessible set G has already been obtained and is given as follows:

$$G = \{O_1,\ O_2,\ O_3,\ \ldots,\ O_{N_o}\} \tag{7}$$

where N_o is the number of operators in G. Now, we can construct the state space model based on the expectation of operators, given the measurement time trace of measurement M and the associated accessible set G. We define the system state vector $\mathbf{x}$ as

$$\mathbf{x} \triangleq (\hat{O}_1,\ \hat{O}_2,\ \hat{O}_3,\ \ldots,\ \hat{O}_{N_o})^T \tag{8}$$

where O_k is the kth operator in G and $\hat{O}_k = \mathrm{Tr}(O_k \rho)$ is the expectation of observable O_k. Note that measurement M can be decomposed onto a linear combination of operators in $\mathbf{x}$ and the time derivative of operators in $\mathbf{x}$ can be obtained using (3). One can obtain the following state space equations:

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{x_0} \\ \mathbf{y} = C\mathbf{x} \end{cases} \tag{9}$$

where A, B, and C are coefficient matrices which may contain some unknown parameters to be identified.

To summarize, we divide the task of deriving a state space model in (9) for a quantum network system into two parts. The first is to find an accessible set so as to define a state vector $\mathbf{x}$. The second is to find the coefficient matrices A, B, and C once the state vector $\mathbf{x}$ is determined. The matrix A can be calculated using (3), B depends on the initial state, C depends on the measurement operators, and $\mathbf{x}_0$ is the initial state [9]. In this article, we mainly focus on the first step since the second step is straightforward after obtaining a proper accessible set.

Although accessible set is important for constructing state space model, the generation of the accessible set is not easy except for systems with simple coupling structures and special measurement schemes. For general cases, the difficulty of generating accessible sets increases rapidly with the number of qubits in the network system. Moreover, note that the ordering of the elements forming the state $\mathbf{x}$ in (8) determines the structure of the matrices A, B, and C. A good ordering should have the following properties:

1) The matrix A has a structure that can simplify further analysis.
2) The matrix A possesses a repetition pattern which is straightforward to extend when the number of qubits in the quantum network system increases.

In this article, we mainly study the generation of accessible sets. Our goal is to simplify the generation processes given in [9] while obtaining a good ordering for elements in the state vector $\mathbf{x}$.

C. Problem Formulation

We assume that the Hamiltonian of a quantum network system consisting of N qubits takes the following form:

$$H = \sum_{k=1}^{N_u} h_k H_k \tag{10}$$

where $\{H_k\}$ are Hermitian operators depending on the way the qubits coupled with each other and $\{h_k\}$ are coupling strengths. N_u is the number of unknown parameters. We call the set

$$\mathcal{F} \triangleq \{H_1,\ H_2,\ H_3,\ \ldots,\ H_k,\ \ldots,\ H_{N_u}\} \tag{11}$$

the *Hamiltonian set* of H.

We generalize M to be a set for all applicable measurement operators as

$$M = \{O_1,\ O_2,\ \ldots\}. \tag{12}$$

Assume the dimension of the system is N_{sys}. Let Λ be an orthonormal basis set of the Lie algebra consisting of all the Hermitian traceless matrices of dimension N_{sys}. The choice of Λ is generally nonunique. But for an n-qubit system, the number of operators in Λ is always $4^n - 1$.

Definition 1: Given the set Λ, for any properly defined $\mathcal{F}$ (M), we can always find a unique set $\bar{\mathcal{F}}$ $(\bar{M})$ such that $\bar{\mathcal{F}}$ $(\bar{M})$ is a minimal basis set of $\mathcal{F}$ (M). We call $\bar{\mathcal{F}}$ $(\bar{M})$ the *decomposed* Hamiltonian (measurement) set.

We set the initial accessible set as $G_0 = \bar{M}$. Then, we iteratively update the accessible set using the following rule until saturated [9]:

$$G_m = [\![G_{m-1}, \bar{\mathcal{F}}]\!] \cup G_{m-1} \tag{13}$$

where

$$[\![G_{m-1}, \bar{\mathcal{F}}]\!] \triangleq \{O_j | \mathrm{Tr}(O_j^\dagger[\tau, \nu]) \neq 0, \exists \tau \in G_{m-1}, \nu \in \bar{\mathcal{F}},\ O_j \in \Lambda\}. \tag{14}$$

Since we require $O_j \in \Lambda$, the accessible set is a linear subset of Λ. See Appendix A for an illustrative example of the generation process of G. The generation rule (14) indicates that finding an accessible set involves finding all of the operators coupled with the measurement operators in (12).

The following definition is used for a concise presentation.

Definition 2: Given a triplet $\{\Lambda, \bar{\mathcal{F}}, \bar{M}\}$, where $\bar{\mathcal{F}}$ is a decomposed Hamiltonian set and $\bar{M}$ is a decomposed measurement set, the function f is defined as $f : \{\Lambda, \bar{\mathcal{F}}, \bar{M}\} \rightarrow G$ where G is the accessible set generated by the triplet $\{\Lambda, \bar{\mathcal{F}}, \bar{M}\}$.

We formulate our problem as follows.

Problem 1: Letting $G = f(\Lambda, \bar{\mathcal{F}}, \bar{M})$ where $\bar{\mathcal{F}}$ is the decomposed Hamiltonian set of (11) and $\bar{M}$ is the decomposed measurement set of (12), we aim to develop an economic method to simplify the generation of the accessible set G with a good ordering according to generation rules (13) and (14).

In Problem 1, the set Λ scales exponentially. Thus, an algorithm can be time-consuming since it may require a full search of Λ, accompanying a high probability to yield an accessible set with an unsatisfactory ordering. Our study aims to investigate Problem 1 for generating a good accessible set efficiently.

III. Main Results

In this section, we first simplify the generation rules (13) and (14) to reduce the computational complexity. We then propose a method to achieve a good ordering for accessible sets. We also provide several lemmas and propositions that can help the calculation.

A. Regarding the Computational Complexity

Define Ω as the set of all operators that are the tensor product of N Pauli matrices and the identity. We have

$$\Omega = \{O | O = \sigma_{i_1} \otimes \sigma_{i_2} \cdots \otimes \sigma_{i_k} \otimes \cdots \sigma_{i_N}\} \tag{15}$$

where $\otimes$ denotes tensor product, $i_k \in \{0, 1, 2, 3\}$, and

$$\begin{aligned} &\sigma_0 := I_{2\times 2}, \qquad \sigma_1 := \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\sigma_2 := \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{16}$$

Note that the identity operator I is not included in Ω. We also have the equality $\sigma_{i_k}^2 = I$.

The set Ω is an unnormalized basis set of the operator space for the network system. For the rest of the article, we work with the set Ω rather than Λ for the generation of accessible sets.

Definition 3: The operation $\lfloor \cdot, \cdot \rceil$ is defined on any operators $A, B \in \Omega$ such that $\lfloor A, B \rceil = O$ where $O \in \Omega$ and $O \propto [A, B]$. If $[A, B] = 0$, the operation $\lfloor A, B \rceil$ returns 0.

The following proposition simplifies the generation of accessible sets for qubit network systems.

Proposition 1: For a qubit network system $\{\Omega, \bar{\mathcal{F}}, \bar{M}\}$, the generation rules (13) and (14) are equivalent to the following rule:

$$G_m = (\!| G_{m-1}, \bar{\mathcal{F}} |\!) \cup G_{m-1} \tag{17}$$

where

$$(\!| G_{m-1}, \bar{\mathcal{F}} |\!) \triangleq \{O_{\tau,\nu} | O_{\tau,\nu} = \lfloor \tau, \nu \rceil, O_{\tau,\nu} \neq 0\ \ \tau \in G_{m-1}, \nu \in \bar{\mathcal{F}}]\}. \tag{18}$$

Proof: The Pauli matrices are orthogonal in the sense

$$\mathrm{Tr}(\sigma_a^\dagger \sigma_b) = \begin{cases} 2 & a = b, \\ 0 & a \neq b \end{cases} \tag{19}$$

where $a, b \in \{0, 1, 2, 3\}$.

Note that the Pauli matrices obey the following commutation relations:

$$[\sigma_a, \sigma_b] = \begin{cases} 2\mathrm{i}\epsilon_{abc}\sigma_c & a \neq b, \\ 0 & a = b \end{cases} \tag{20}$$

where $a, b, c \in \{1, 2, 3\}$ and the constant ϵ_{abc} is the Levi-Civita symbol. Equation (20) indicates that the commutator of Pauli matrices yields either a matrix that is proportional to another Pauli matrix or 0. Based on this fact, we have

$$\mathcal{C}_{O_{1,2}}[O_1, O_2] \in \Omega \quad \forall O_1, O_2 \in \Omega$$

where $\mathcal{C}_{O_{1,2}}$ is a proper nonzero coefficient. We can conclude that there exists a proper nonzero coefficient $\mathcal{C}_{\tau,\nu}$ such that

$$\mathcal{C}_{\tau,\nu}[\tau, \nu] \in \Omega \quad \forall \tau \in G_0, \nu \in \mathcal{F}. \tag{21}$$

Equation (21) indicates that the accessible set $G_m \subset \Omega$ given that $G_{m-1} \subset \Omega$ using generation rule (18). In our case, we have $\bar{\mathcal{F}} \subset \Omega$ and $\bar{G}_0 \subset \Omega$, which assures that the generation rule (18) can guarantee that $G \subset \Omega$. Letting

$$O_{\tau,\nu} = \mathcal{C}_{\tau,\nu}[\tau, \nu] \tag{22}$$

we have $O_{\tau,\nu} \in \Omega$, which confirms that the commutator of two operators in Ω yields another operator that is proportional to an operator in Ω. According to (20), if $O_{\tau,\nu} \neq 0$, then

$$\mathrm{Tr}(O^{\dagger}_{\tau,\nu}[\tau, \nu]) \neq 0. \tag{23}$$

According to (19), for any $O \in \Omega$ with $O \neq O_{\tau,\nu}$, we have

$$\mathrm{Tr}(O^{\dagger}[\tau, \nu]) = 0. \tag{24}$$

Equations (23) and (24) together indicate that $O_{\tau,\nu}$ is the operator that satisfies the requirement in (14) and, thus, should be added into the accessible set. The generation rule (14) can thus be simplified to (18). ■

Proposition 1 indicates that all of the nonzero commutators of the operators in a former accessible set G_{m-1} and the operators in $\bar{F}$ should be added into the accessible set G_m. Compared with (14), (18) avoids a full search of the elements in Ω. Using (14), the average computation complexity of finding a single element in the set Ω is $O(4^N 2^{3N})$. Using (18), the computational complexity of updating an element is reduced to $O(2^{3N})$.

Problem 1 can now be restated as the following problem with a lower computational complexity.

Problem 2: Develop an economic method to generate the accessible set $G = f(\Omega, \bar{F}, \bar{M})$ with a good ordering, using rules (17) and (18).

B. Graphs Generated by Accessible Sets

Graphs can be employed to demonstrate the generation of accessible sets. We benefit from graphs mainly in three aspects. First, a graph visualizes the relationship between operators in the corresponding accessible set. Moreover, the repetition pattern revealed by a graph when generating an accessible set has the potential to be summarized and used to extend an accessible set to any given qubit number. Second, graphs can be used to arrange the ordering of element operators in the state vector to achieve a good structure of the state space matrices. Third, graphs can help with the proofs of our lemmas and propositions.

We assign each accessible set G a graph $\mathbb{G}$. The vertices of $\mathbb{G}$ are the elements in the corresponding accessible set G. We say there is an edge between two vertices O_m and O_n if and only if there exists $\nu \in \bar{F}$ such that

$$\mathrm{Tr}(O^{\dagger}_{n}\lfloor O_m, \nu\rceil) \neq 0. \tag{25}$$

Here we prove that such an edge (if it exists) is unique. For Pauli matrices, we observe that if $\lfloor \sigma_m, \sigma_i \rceil = \sigma_n$ and $\lfloor \sigma_m, \sigma_j \rceil = \sigma_n$ where $i, j, m, n \in \{0, 1, 2, 3\}$, then $\sigma_i = \sigma_j$. Thus, given $\nu, u \in \bar{F}$, if

$$\begin{cases} \mathrm{Tr}(O^{\dagger}_{n}\lfloor O_m, \nu\rceil) \neq 0, \\ \mathrm{Tr}(O^{\dagger}_{n}\lfloor O_m, u\rceil) \neq 0, \end{cases} \text{for some } O_m, O_n \in \Omega \tag{26}$$

we have

$$\nu = u. \tag{27}$$

Hence, there are no multiple edges with the same direction between any two vertices. Therefore, we can use ν to label the edge $\langle O_m, O_n\rangle$ and ν is called the *edging operator*. Furthermore, we prove that there is no loop in $\mathbb{G}$, and, thus, $\mathbb{G}$ is a simple graph. Note that we always have

$$\mathrm{Tr}(O^{\dagger}_{m}\lfloor O_m, \nu\rceil) = 0 \tag{28}$$

for any $O_m, \nu \in \Omega$. This means there exists no edge $\langle O_m, O_m\rangle$ and, thus, there is no loop in the graph. We conclude that all the graphs associated with accessible sets defined in this article have no loops or multiple edges, which means they are simple graphs.

Labeling the vertices of graph $\mathbb{G}$ with natural numbers, we obtain the adjacency matrix $\mathbb{A}$ whose (i, j)th entry is 1, if and only if there is an edge connecting the ith and jth vertices [37]. The state space matrix A in (9) has the same structure as $\mathbb{A}$, while having different elements from $\mathbb{A}$. The graph and the matrix A share the same pattern in a certain sense. Thus, studying on the graph provides insights on the accessible set and the corresponding state space representation. In the following, we provide more features of the graph.

The graph $\mathbb{G}$ can be described as $\mathbb{G} = \{G, \mathbb{E}\}$, where the accessible set G is a set of vertex operators and $\mathbb{E}$ is the set of all of the edges. Moreover, we have the following definition.

Definition 4: A path in the graph can be specified by a set of vertex operators $(O_1, O_2, \ldots, O_m)$ or equivalently by the starting operator, ending operator and a sequence of edging operators $\{O_1, (\nu_1, \nu_2, \cdots), O_m\}$ where $\nu_i = \langle O_i, O_{i+1}\rangle$. We refer to the sequence $E = (\nu_1, \nu_2, \ldots)$ as an *edging sequence* which is a sequence of edging operators. We define $S(\bar{F})$ as the set of all the sequences of finite edging operators chosen from $\bar{F}$. We also define $C(E) \triangleq \underbrace{\nu_1, \nu_2, \ldots}$ as the collection of edging operators in E.

Remark 2: The notation $E \in S(\bar{F})$ indicates that all of the elements in E belong to $\bar{F}$. The reason that the triplet $\{O_1, E, O_m\}$ can specify a path is based on the fact that the graphs in this article are all simple graphs. It is worth noting the differences between a set, a collection, and a sequence. Sets and sequences can be regarded as specific class collections that are endowed with different features. While the uniqueness of objects in a collection is not guaranteed, a set is defined as a collection of distinct objects. While objects in a collection may not be ordered, elements in a sequence are uniquely ordered. For example, while $E_1 = (X, Y, Y)$ and $E_2 = (Y, X, Y)$ are two different sequences, the collections $C^1_E = C(E_1) = \underbrace{X, Y, Y}$ and $C^2_E = C(E_2) = \underbrace{Y, X, Y}$ are the same. Moreover, we have $E_1, E_2 \in S(\{X, Y\})$ which indicates that sets of edging operators forming the sequences E_1 and E_2 are the same.

A graph is called undirected if there is no direction assigned to the edges. We have the following lemma which states that any graph generated by an accessible set is essentially undirected.

Lemma 1: Assume that $\mathbb{G} = \{G, \mathbb{E}\}$ where $G = f(\Omega, \bar{F}, \bar{M})$ and $\mathbb{E}$ is the corresponding set of edges. Then each edge of $\mathbb{G}$ is bi-directed if endowed with direction.

Proof: Suppose O_m and O_n are two different vertices and there is an edge $\langle O_m, O_n \rangle$ connecting O_m and O_n. We prove that there exists an edge $\langle O_n, O_m \rangle$ and it has the same label as $\langle O_m, O_n \rangle$.

According to the definition of an edge and the fact that we have an edge $\langle O_m, O_n \rangle$, there exists a $\nu \in \bar{F}$ such that

$$O_n = \lfloor O_m, \nu \rceil. \tag{29}$$

Then the edge $\langle O_m, O_n \rangle$ is labeled by ν. According to (20), we have

$$O_m = \lfloor O_n, \nu \rceil. \tag{30}$$

Then the edge $\langle O_n, O_m \rangle$ is also labeled by ν. Hence, each edge of G is bi-directed if endowed with direction. To put it differently, the iterative rules given in (13) and (18) can achieve a bi-directional search. ■

Considering Lemma 1, direction becomes a trivial property for graphs representing accessible sets. Hence, we regard all graphs employed in this article to be undirected.

Note that a graph is connected if there exists at least one path between every pair of vertices. An *induced subgraph* of a graph is another graph, formed from a subset of the vertices of the graph and all of the edges connecting pairs of vertices in that subset. We have the following lemma.

Lemma 2: Let $\mathbb{G} = \{G, \mathbb{E}\}$ where $G = f(\Omega, \bar{F}, \bar{M})$ and $\mathbb{E}$ is the corresponding set of edges. Also let $\mathbb{M} = \{\bar{M}, \mathbb{E}_M\}$ be an induced graph of $\mathbb{G}$ where all of the vertices of $\mathbb{M}$ are in the measurement set $\bar{M}$ and $\mathbb{E}_M$ is the corresponding set of edges. If the graph $\mathbb{M}$ is connected, then the graph $\mathbb{G}$ is connected.

Proof: All of the elements in the accessible set G are generated by the elements in the initial set $\bar{M}$. Thus, they are connected with the elements in $\bar{M}$ according to the definition of the graph $\mathbb{G}$. Since $\mathbb{M}$ is assumed to be connected, the graph $\mathbb{G}$ is also connected. ■

Lemma 3: Given $G = f(\Omega, \bar{F}, \bar{M})$ and $\tilde{G} = f(\Omega, \bar{F}, \tilde{M})$ where $\tilde{M}$ is a nonempty subset of G. If G is connected, we have $\tilde{G} = G$.

Proof: Since we suppose that an undirected graph G is connected, the accessible set can be obtained starting from an arbitrary group of operators (not necessary the measurement operators) that belong to the accessible set, using the generation rules (17) and (18). Then Lemma 3 follows. ■

C. Special Consideration for a Class of Spin Chain Systems

A chain system, where qubits are connected in the form of a string, is a fundamental and typical quantum network system (see Fig. 2) [10], [11]. Here, we consider a chain system consisting of N qubits. The system Hamiltonian is

$$H = \sum_{k=1}^{N-1} h_k (X_k X_{k+1} + Y_k Y_{k+1}) \tag{31}$$

where the following notation is used $X := \sigma_x$, $Y := \sigma_y$, and $Z := \sigma_z$. The subscript k indicates that the operator is on the kth qubit. To write the operators in a compact form, we omit the tensor product symbol and the identity operator unless otherwise specified. Hence, for example, we use the simplified notation $X_k X_{k+1}$ to represent

$$I^{\otimes(k-1)} \otimes X_k \otimes X_{k+1} \otimes I^{\otimes(N-k-1)}. \tag{32}$$

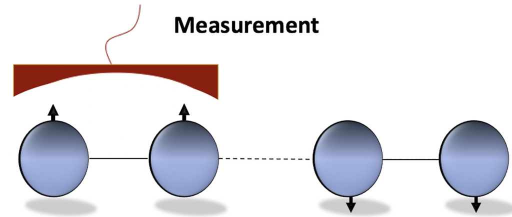

Fig. 2. Example of a quantum network system whose elements are qubit systems coupled in the form of a chain. The measurement is on the first several (two in this example) qubits in the chain system.

The system whose Hamiltonian is given in (31) is an exchange model without transverse field [38], [39]. The coupling Hamiltonian between the kth and $(k+1)$th qubit is $h_k(X_k X_{k+1} + Y_k Y_{k+1})$. The decomposed set $\bar{F}$ for the chain system in (31) is

$$\bar{F} = \{X_1 X_2,\ Y_1 Y_2,\ \ldots,\ X_k X_{k+1},\ Y_k Y_{k+1},\ \ldots,\} \tag{33}$$

where $1 \le k \le N-1$.

We present the following proposition to help with the generation of accessible sets for the system with Hamiltonian given in (31).

Proposition 2: Given $\bar{F}$ as in (33) and the measurement set $\bar{M} = \{Z^{\otimes(m-1)} X_m\}$ where m is an arbitrary integer satisfying $1 \le m \le N$, we have

$$G^X := f(\Omega, \bar{F}, \bar{M}) = \{O_1^X,\ \ldots,\ O_k^X,\ \ldots\} \tag{34}$$

where

$$O_k^X = \begin{cases} Z^{\otimes(m+k-1)} X_{m+k}, & k \text{ is even,} \\ Z^{\otimes(m+k-1)} Y_{m+k}, & k \text{ is odd} \end{cases} \tag{35}$$

and $0 \le k \le N-m$. Similarly, if the measurement set is given as $\bar{M} = \{Z^{\otimes(m-1)} Y_m\}$, the corresponding accessible set is

$$G^Y := f(\Omega, \bar{F}, \bar{M}) = \{O_1^Y,\ \ldots,\ O_k^Y,\ \ldots\} \tag{36}$$

where

$$O_k^Y = \begin{cases} Z^{\otimes(m+k-1)} Y_{m+k}, & k \text{ is even,} \\ Z^{\otimes(m+k-1)} X_{m+k}, & k \text{ is odd} \end{cases} \tag{37}$$

and $0 \le k \le N-m$.

Proof: According to (18), the iterative generation rule involves adding nonzero operators that are generated by taking the commutator operation on operators in G_{m-1} and operators in $\bar{F}$ into the new accessible set G_m. Here we find the following common patterns:

$$\begin{aligned}
&[O_{(1,k-1)}Z_kY_{k+1}, X_{k+1}X_{k+2}] \\
&\quad = O_{(1,k-1)}Z_k[Y_{k+1}, X_{k+1}]X_{k+2} \\
&\quad = (-2\mathrm{i})O_{(1,k-1)}Z_kZ_{k+1}X_{k+2} \\
&[O_{(1,k-1)}Z_kX_{k+1}, Y_{k+1}Y_{k+2}] \\
&\quad = O_{(1,k-1)}Z_k[X_{k+1}, Y_{k+1}]Y_{k+2} \\
&\quad = (2\mathrm{i})O_{(1,k-1)}Z_kZ_{k+1}Y_{k+2} \\
&[O_{(1,k-1)}Z_kY_{k+1}, Y_{k+1}Y_{k+2}] \\
&\quad = O_{(1,k-1)}Z_k[Y_{k+1}, Y_{k+1}]X_{k+2} = 0 \\
&[O_{(1,k-1)}Z_kX_{k+1}, X_{k+1}X_{k+2}] \\
&\quad = O_{(1,k-1)}Z_k[X_{k+1}, X_{k+1}]Y_{k+2} = 0
\end{aligned} \tag{38}$$

where $O_{(1,k-1)}$ is an operator acting on the first $(k-1)$ operators. For a system whose Hamiltonian takes the form of (31), $O_{(1,k-1)}Z_kY_{k+1} \in G$ where $1 \leq k \leq N-2$ leads to $O_{(1,k-1)}Z_kZ_{k+1}X_{k+2} \in G$. If we have $O_{(1,k-1)}Z_kX_{k+1} \in G$ where $1 \leq k \leq N-2$, then we also have $O_{(1,k-1)}Z_kZ_{k+1}Y_{k+2} \in G$. The equalities in (38) provide us with operators that should be added when all of the operators in G can be written either in the form of $O_{(1,k-1)}Z_kY_{k+1}$ or in the form of $O_{(1,k-1)}Z_kX_{k+1}$. Note that the added operators $O_{(1,k-1)}Z_kZ_{k+1}X_{k+2}$ and $O_{(1,k-1)}Z_kZ_{k+1}Y_{k+2}$ can be written in the form $O_{(1,k)}Z_{k+1}X_{k+2}$ and $O_{(1,k)}Z_{k+1}Y_{k+2}$, which facilitates the iterative generation of accessible sets. ∎

Proposition 2 provides us with accessible sets for cases such as 1), 3), and 5) in Section IV.

D. Improving the Ordering

The results in Section III-A concern the reduction of computational complexity. Here, we focus on the generation of accessible sets with good ordering. Two main objectives are as follows:

1) to find a repetition pattern for the state space model as the number of nodes increases;
2) to reveal the connections between element operators in G.

These two objectives are vital for finding a repetition pattern for the state space model and writing down an N-qubit system model for arbitrary N. Otherwise, one only has accessible sets for several limited values of N, and the identification, analysis, and control of the system will be difficult to be extended. Arranging the order of element operators in the state vector according to the graph, it is likely to obtain a state space model with good structure.

Definition 5: We denote the set $\mathcal{B} = \{I_{2\times 2}, \sigma_x.\sigma_y, \sigma_z\}$ as the cell set and an operator $O \in \mathcal{B}$ is a cell operator.

In this article, we use the notation $X := \sigma_x$, $Y := \sigma_y$, and $Z := \sigma_z$ interchangeably so that the cell set can also be written as $\mathcal{B} = \{I_{2\times 2}, X, Y, Z\}$.

Definition 6: A set G is said to be *k-finite* if every operator $O \in G$ takes the following form:

$$O = \sigma_{s_1^k} \otimes \sigma_{s_2^k} \otimes \sigma_{s_3^k} \otimes \cdots \otimes \sigma_{s_M^k} \tag{39}$$

where $M < \infty$ is the number of cell operators in O, and

$$s_j^k \in \begin{cases} \{0,1,2,3\}, & 1 \leq j < k \\ \{1,2,3\}, & j = k \\ \{0\}, & k < j \leq M. \end{cases} \tag{40}$$

We start from an N-qubit chain system with a Hamiltonian as in (31). For such a system, we have the following proposition.

Proposition 3: For an i-qubit network system with the Hamiltonian given in (31), $\bar{F}_i$ given in (33) and $\bar{M}$ connected, let $G_i = f(\Omega, \bar{F}_i, \bar{M})$ be the corresponding accessible set. Define a series of sets $G_{\lfloor k}$ $(1 \leq k \leq i)$ as

$$G_{\lfloor k} = \begin{cases} G_k, & k = 1 \\ G_k - G_{k-1}, & k > 1. \end{cases} \tag{41}$$

The symbol "$-$" between any sets A and B as $A - B$ indicates the subtraction of the set B from the set A. We have the following assertions:

Assertion 1: $\forall 1 \leq l \leq \mu \leq i, G_l \subseteq G_\mu$.

Assertion 2: The set $G_{\lfloor k}$ is k-finite.

Assertion 3: There exists $\bar{F}_k \subset \bar{F}$ such that $G_{\lfloor k} = f(\Omega, \bar{F}_k, \{O_k\})$ where O_k can be any operator in $G_{\lfloor k}$ and $\bar{F}_k$ can be independent of the choice of O_k.

Proposition 3 reveals the relations between the sets G_k and $G_{\lfloor k}$ for $k = 1, 2, \ldots$. Please see Appendix B for the proof.

Equation (41) is equivalent to $G_i = G_{k-1} \cup G_{\lfloor k}$, which means one only needs to find $G_{\lfloor k}$ to obtain the accessible set G_i given the accessible set G_{k-1} for a class of spin chain systems. Moreover, if we observe a pattern shared by all of the graphs $\mathbb{G}_{\lfloor k}$, one can generate the accessible set G_n for any given n. Furthermore, Assertion 3 in Proposition 3 confirms that all of the induced subgraphs $\mathbb{G}_{\lfloor k}$ are connected. The connectivity of $\mathbb{G}_{\lfloor k}$ indicates that all of the subsets $G_{\lfloor k}$ can be generated by starting from an arbitrary operator that belongs to $G_{\lfloor k}$. After finding an arbitrary operator $O \in G_{\lfloor k}$, one can obtain all of the operators in $G_{\lfloor k}$.

We want to design a search algorithm that is suitable for generating all of the subsets $G_{\lfloor k}$. In the set G, we place the elements of $G_{\lfloor k}$ in front of the elements of $G_{\lfloor k+1}$. For different systems and measurement schemes, one needs to design a proper search rule accordingly. The main idea employed in generating an accessible set with a good ordering is to divide the accessible set G into subsets to reveal a generation pattern that is shared by the accessible sets as the number of qubits increases.

Here, we summarize the generation process. Given a measurement scheme, we first decompose the measurement set and the Hamiltonian set into the form we defined in Definition 1. Then we observe the measurement set to see if Proposition 2 can be applied to this situation. For some cases, we can obtain an accessible set at this stage. Otherwise, we determine if the graph associated with the accessible set is connected or not. If the graph is connected, we divide the accessible set into subsets to find certain repetition patterns when generating the subsets. If the graph associated with an accessible set is not connected, this article can still provide some insight. Generally, a graph can be divided into several connected subgraphs. The ideas in this article can thus still be applied for the generation of the connected subgraphs. Collecting all of the vertices of the subgraphs together provides a complete accessible set.

IV. Illustrative Examples

Here we present several examples to demonstrate the generation of a proper accessible set with good ordering. The object system is a chain system consisting of N qubits. The system Hamiltonian is given in (31) and the set $\bar{F}$ is given in (33). We provide accessible sets for the following six measurement schemes:

1) $M = \{X_1\}$
2) $M = \{Z_1\}$
3) $M = \{Z_1Y_2\}$
4) $M = \{Y_1Z_2\}$
5) $M = \{Z_1Z_2X_3\}$
6) $M = \{X_1Y_2Z_3\}$.

For cases 1) and 2), only the first qubit in the chain system is measured. For cases 3) and 4), we measure the first two qubits of the chain system. For cases 5) and 6), the first three qubits are measured. These cases cover most of the common fundamental measurement settings, and several similar settings are omitted. For example, from the analysis on case 1), one can straightforwardly write down the analysis result when the measurement is $M = \{Y_1\}$. After we obtain the accessible sets for these cases, we can establish the corresponding state space models using (3) and (9). Based on these results, further tasks can be performed, like analyzing the Hamiltonian identifiability and developing effective Hamiltonian identification algorithms (see [11] and [17] for specific examples).

To visualize the generation process, we employ graphs to describe accessible sets. According to Proposition 3, when $\bar{M}$ has only one element, the graph $\mathbb{G}$ generated by a complete accessible set G is connected, which means there is always a path connecting any two operators in G. This holds for all of the examples in this section and is clearly exemplified by case 2) (see Fig. 3). The graph $\mathbb{G}_{\lfloor k}$ associated with subset $G_{\lfloor k}$ is also connected under the assumption in Proposition 3. This can also be observed from all of the examples, especially from cases 2), 4), and 6).

For cases 1), 3), and 5), we present analytical formula for the accessible set for an arbitrary integer N. For cases 2), 4), and 6), we present the generation of the accessible set for a fixed qubit number N, employing graphs to find the repetition pattern generating the accessible set. By observing and summarizing those generation patterns, we can determine the accessible set for any given N. Arranging the elements according to the graphs can provide us with a good structure for the state space equation matrices A, B, and C in (9).

1) **Measuring $\mathbf{X_1}$**. According to Proposition 2, the accessible set G can be obtained immediately as

$$G = \begin{cases} \{X_1, Z_1Y_2, Z_1Z_2X_3, \cdots, Z^{\otimes(N-1)}Y_N\}, & N \text{ is even,} \\ \{X_1, Z_1Y_2, Z_1Z_2X_3, \cdots, Z^{\otimes(N-1)}X_N\}, & N \text{ is odd.} \end{cases} \tag{42}$$

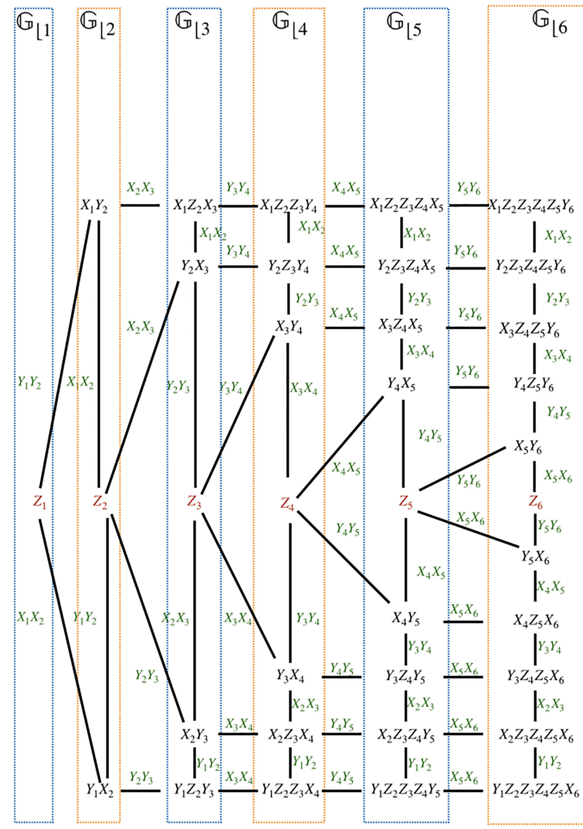

Fig. 3. Accessible set G when measuring Z_1. The system contains six qubits. Element operators (marked in black and red) are vertices of the graph. Edges connecting vertices are labeled by operators (marked in green) in the set $\bar{F}$ used to generate the vertices.

2) **Measuring $\mathbf{Z_1}$**. We have the following iterative generation rule:

$$\begin{cases} \lfloor Z_k, X_kX_{k+1}\rceil = Y_kX_{k+1} \\ \lfloor Y_kX_{k+1}, Y_kY_{k+1}\rceil = Z_{k+1}. \end{cases} \tag{43}$$

From (43) and the fact that Z_1 is in the accessible set, it can be identified that the operators Z_k, where $1 \le k \le N$, are all in the accessible set G.

Aiming to find all of the other operators in the accessible set, we divide the accessible set G into the following subsets:

$$G = \cup_{k=1}^{N} G_{\lfloor k} = \cup_{k=1}^{N} \{Z_k, \ldots\} \tag{44}$$

where the subset $G_{\lfloor k}$ is k-finite.

We denote Z_k as the "core" operator in the subset $G_{\lfloor k}$. A "core" operator is an operator selected from $G_{\lfloor k}$ and serves as the starting operator while generating $G_{\lfloor k}$. Since the graph $G_{\lfloor k}$ is connected, one can select any operator in $G_{\lfloor k}$ to be a core operator according to Proposition 3, which means that all of the other operators in $G_{\lfloor k}$ can be generated from Z_k by rule (18).

In Fig. 3, the accessible set is given for the case where there are six qubits in the network system. Starting from the core operator, the generation of the operators in G forms a graph which follows a clear repetition pattern. In subset $G_{\lfloor 1}$ (in the blue dashed box), there is only one operator Z_1 which is the measurement operator. In subset $G_{\lfloor 2}$ (in the yellow dashed box), there are three

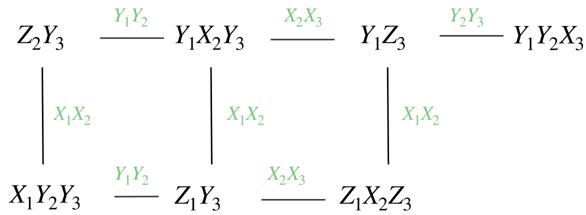

Fig. 4. Generation procedure for operators that form the set $G_{\lfloor 3}$.

operators Z_2, X_1Y_2, and Y_1X_2. Following the special pattern revealed in Fig. 3, one can generate an accessible set for a chain system with an arbitrary number of qubits. Moreover, we can also turn to Fig. 3 for a good ordering when constructing the system state variable $\boldsymbol{x}$.

3) **Measuring $\mathbf{Z_1Y_2}$** . Given the initial measurement operator Z_1Y_2, the accessible set is as follows according to Proposition 2:

$$G = \{O_1, O_2, \ldots O_k, \ldots\} \tag{45}$$

where

$$O_k = \begin{cases} Z_1Z_2\cdots Z_{k-1}Y_k, & k \text{ is even,} \\ Z_1Z_2\cdots Z_{k-1}X_k, & k \text{ is odd.} \end{cases} \tag{46}$$

4) **Measuring $\mathbf{Y_1Z_2}$** . We have the following equality:

$$\begin{cases} \lfloor Y_1Z_2, Y_1Y_2\rceil = X_2 \\ \lfloor X_2, Y_2Y_3\rceil = Z_2Y_3 \end{cases} \tag{47}$$

which indicates that the operator Z_2Y_3 is in the accessible set G. Therefore, from Proposition 2, the following operators are in the accessible set G:

$$\{X_2,\ Z_2Y_3,\ Z_2Z_3X_4,\ \ldots,\ I\otimes Z^{\otimes(k-2)}O_k,\ \ldots\} \subset G \tag{48}$$

where $3 \le k \le N$ and

$$O_k = \begin{cases} X & k \text{ is even,} \\ Y & k \text{ is odd.} \end{cases} \tag{49}$$

Aiming to find all of the other operators in G, we divide it into the following subsets:

$$\begin{aligned} G &= G_{\lfloor 2} \cup G_{\lfloor 3} \cup G_{\lfloor 4} \cup \cdots \cup G_{\lfloor k} \cdots \\ &= \{X_2,\ Y_1Z_2\} \cup \{Z_2Y_3,\ \cdots\} \cup \{Z_2Z_3X_4,\ \cdots\} \cup \cdots \end{aligned} \tag{50}$$

where the subset $G_{\lfloor k}$ is k-finite.

Let the "core" operator of $G_{\lfloor k}$ be O_k^c as

$$O_k^c = \begin{cases} Z_2\cdots Z_{k-1}X_k, & k \text{ is even,} \\ Z_2\cdots Z_{k-1}Y_k, & k \text{ is odd.} \end{cases} \tag{51}$$

According to Proposition 3, all of the other operators in $G_{\lfloor k}$ can be generated from O_k^c by rule (18) given that the core operator O_k^c belongs to the subset $G_{\lfloor k}$.

The subset $G_{\lfloor 2} = \{Y_1Z_2,\ X_2\}$ only contains two operators. Starting from the core operator Z_2Y_3, elements in $G_{\lfloor 3}$ can be inferred and the generation procedure is shown in Fig. 4. For example, given that $Z_2Z_3X_4 \in G_{\lfloor 4}$ and $[Z_2Z_3X_4, Y_1Y_2] = -2\mathrm{i}Y_1X_2Z_3X_4$, it can be inferred that $Y_1X_2Z_3X_4 \in G_{\lfloor 4}$ as well, according to (18). Similarly, the generating processes and element operators for the subsets $G_{\lfloor 4}$ and $G_{\lfloor 5}$ are shown in Figs. 5 and 6, respectively. The generation patterns for those sets are similar and repetitive. It can be seen that the number of elements in $G_{\lfloor 3}$ is $3+4=7$; the number of elements in $G_{\lfloor 4}$ is $4+5+6=15$; the number of elements in $G_{\lfloor 5}$ is $5+6+7+8=26$. Using the induction method, the number of operators in $G_{\lfloor k}$ is $(3k-2)(k-1)/2$.

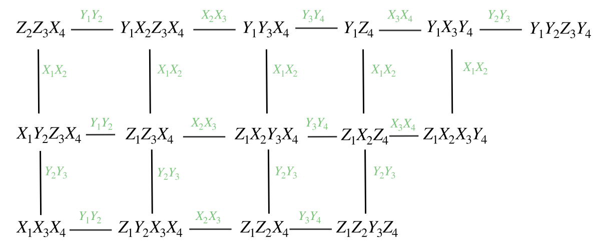

Fig. 5. Generation procedure for operators forming the set $G_{\lfloor 4}$. The operators in black are in the accessible set while operators in green are in $\bar{F}$.

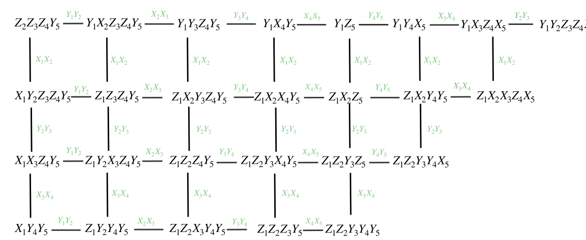

Fig. 6. Generation procedure for operators in set $G_{\lfloor 5}$.

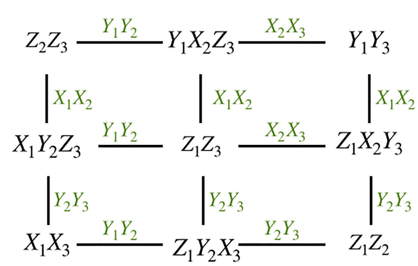

Fig. 7. Accessible set when measuring $X_1Y_2Z_3$. The number of qubits is 3.

For an N-qubit chain system, the total number of operators in the accessible set G is

$$|G| = \sum_{K=2}^{N} \frac{(3k-2)(k-1)}{2} = \frac{N^3 - N^2}{2}. \tag{52}$$

From the analysis above, it is clear that the number of operators in G scales as N^3, which can be far more than the qubit number.

5) **Measuring $\mathbf{Z_1Z_2X_3}$** . Given the initial measurement operator $Z_1Z_2X_3$, the accessible set is as follows according to Proposition 2:

$$G = \{O_1,\ O_2,\ \ldots,\ O_k,\ \ldots\} \tag{53}$$

where

$$O_k = \begin{cases} Z_1Z_2\cdots Z_{k-1}Y_k, & k \text{ is even,} \\ Z_1Z_2\cdots Z_{k-1}X_k, & k \text{ is odd.} \end{cases} \tag{54}$$

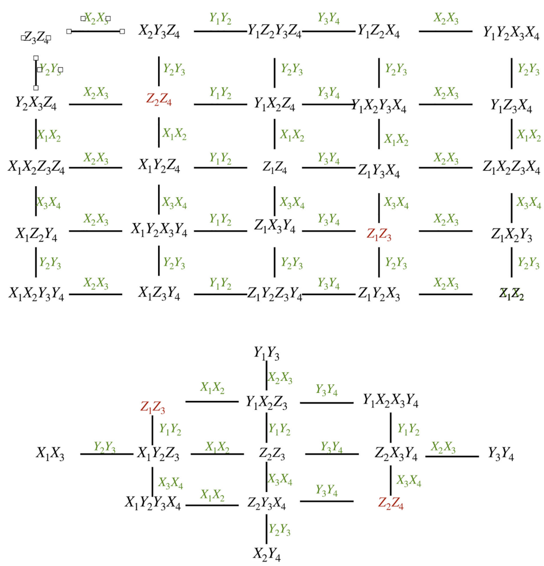

Fig. 8. Accessible set when measuring $X_1Y_2Z_3$. The number of qubits is 4. The graph generated by G_4 cannot be presented in an uncrossed two-dimensional graph. Hence, we separate it into two graphs. However, there are repetitive elements in the two graphs (see the ones marked in red), which means the complete graph is still a connected simple graph.

6) **Measuring $\mathbf{X_1Y_2Z_3}$** . When the measurement operator is $X_1Y_2Z_3$, the accessible set G for 3-qubit and 4-qubit systems are shown in Figs. 7 and 8, respectively. In both situations, the graphs show some complicated generation patterns with a certain repetition mode.

V. Conclusion

To model a quantum system using state space equations for the purpose of parameter identification, one needs to define a proper state vector. Mathematically, one can define the state vector using any complete set of bases. However, the dimension of the state space model is often extremely high for a quantum network system. This problem can be solved or mitigated by defining the state as a vector of expectations of element operators in a reduced accessible set associated with the measurement. By generating a proper accessible set which only contains minimal number of operators coupled with the measurement, the model dimension may be reduced dramatically. Moreover, we may obtain coefficient matrices with good properties by ordering the element operators in the accessible set in a proper sequence. Thus, a well-generated accessible set can be used to simplify further analysis for, e.g., Hamiltonian identification.

Based on the above motivation, we investigated the generation of accessible sets, given a system Hamiltonian and a measurement operator. We obtained a series of results that can simplify the generation procedure for accessible sets for a class of network systems. We also introduced graphs to demonstrate the generation of accessible sets and to guide the ordering of elements in the state space vectors. Several examples were presented where the accessible sets for different measurement schemes were obtained, demonstrating the effectiveness of our method.

Appendix A
Example to Demonstrate the Generation Rules (13) and (14)

For a 2-qubit system, we choose the basis set Λ as

$$\Lambda = \{X_1,\ Y_1,\ Z_1,\ X_2,\ Y_2,\ Z_2,\ X_1X_2,\ X_1Y_2,\ X_1Z_2,\ Y_1X_2,\ Y_1Y_2,\ Y_1Z_2,\ Z_1X_2,\ Z_1Y_2,\ Z_1Z_2\}. \tag{55}$$

Assume the system Hamiltonian is $H = \frac{1}{2}(X_1X_2 + Y_1Y_2)$ which yields the decomposed set $\bar{F} = \{X_1X_2,\ Y_1Y_2\}$. Select the measurement as $\bar{M} = \{Y_1Z_2\}$ which is already decomposed. The generation procedure is as follows.

1) Set $G_0 = \bar{M} = \{Y_1Z_2\}$.
2) Then we calculate $[\![G_0, \bar{F}]\!]$.
   a) Let $\tau = Y_1Z_2$ and $\nu = X_1X_2$. Since $[\tau, \nu] = 0$, no operators need to be added into the set.
   b) Let $\tau = Y_1Z_2$ and $\nu = Y_1Y_2$. Since $[\tau, \nu] = -2\mathrm{i}X_2$, there is only one operator $X_2 \in \Lambda$ that satisfies $\mathrm{Tr}(X_2^\dagger[\tau, \nu]) \neq 0$. Thus, the operator X_2 is added into $[\![G_0, \bar{F}]\!]$.
3) The set $G_1 = [\![G_0, \bar{F}]\!] \cup G_0 = \{Y_1Z_2,\ X_2\}$. Since $G_1 \neq G_0$ which means the generation is not saturated, the algorithm continues.
4) Then we calculate $[\![G_1, \bar{F}]\!]$.
   a) Let $\tau = X_2$ and $\nu = X_1X_2$. Since $[\tau, \nu] = 0$, no operators need to be added into the set.
   b) Let $\tau = X_2$ and $\nu = Y_1Y_2$. Since $[\tau, \nu] = 2\mathrm{i}Y_1Z_2$, there is only one operator $Y_1Z_2 \in \Lambda$ that satisfies $\mathrm{Tr}((Y_1Z_2)^\dagger[\tau, \nu]) \neq 0$. Thus, operator Y_1Z_2 should be added into $[\![G_0, \bar{F}]\!]$.
5) The set $G_2 = [\![G_1, \bar{F}]\!] \cup G_1 = \{Y_1Z_2,\ X_2\} \cup \{Y_1Z_2\} = \{Y_1Z_2,\ X_2\}$. Since $G_2 = G_1$ indicates that the generation process is saturated, the generation process stops and the final accessible set is $G = \{Y_1Z_2,\ X_2\}$.

Appendix B
Proof of Proposition 3

In order to prove Proposition 3, we first present preliminaries and several lemmas that will be used. The following two definitions are given first.

Definition 7: Given $G = f(\Omega, \bar{F}, \bar{M})$ and its corresponding graph $\mathbb{G}$, two vertices $O_m \in G$ and $O_n \in G$ are called adjacent if and only if there exists $\nu \in \bar{F}$ such that

$$\mathrm{Tr}(O_n^\dagger, [O_m, \nu]) \neq 0 \tag{56}$$

which means that there is an edge (labeled by ν) connecting vertices O_m and O_n in the graph $\mathbb{G}$.

Definition 8: Graph $\mathbb{G}_m$ and graph $\mathbb{G}_n$ are adjacent if and only if there exists at least one vertex in $\mathbb{G}_m$ that is adjacent to a vertex in $\mathbb{G}_n$.

We plan to use induction. To simplify the narrative, we divide elements in $G_{\lfloor k+1}$ into two classes $G^1_{\lfloor k+1}$ and $G^2_{\lfloor k+1}$ such that we have the following:

1) $G^1_{\lfloor k+1} \subset G_{\lfloor k+1}$ is the set of operators that are adjacent to elements in G_k;

2) $G^2_{\lfloor k+1} = G_{\lfloor k+1} - G^1_{\lfloor k+1}$ is the set of operators that are not adjacent to any element in G_k.

Note that the following three statements are equivalent: 1) $G^1_{\lfloor k+1}$ can be generated by the triplet $\{\Omega, \bar{F}_k, \{O_{k+1}\}\}$ where O_{k+1} is an arbitrary operator that belongs to $G^1_{\lfloor k+1}$; 2) the graph $\mathbb{G}^1_{\lfloor k+1}$ is connected; 3) there is always a path between every pair of vertices in the graph $\mathbb{G}^1_{\lfloor k+1}$. Thus, the other two statements are proved if any one of the statements are proved, and we may use the three statements in the following proofs interchangeably.

Note that an arbitrary operator in Ω is formed by the tensor product of a sequence of cell operators in $\mathcal{B} = \{I_{2\times 2}, X, Y, Z\}$. For a given operator $O \in \Omega$, we refer to the cell operator on the jth qubit as the jth cell operator. Moreover, for an operator $O \in G_k$ where G_k is k-finite, we refer to the kth cell operator as the *ending operator*. For example, the third cell operator of $O = X \otimes Y \otimes I \otimes Z$ is I and the ending operator of O is Z, given that $O \in G_4$. We can also say that the operator O ends with Z. Now we give five lemmas (the proof details can be found in [40]).

Lemma 4: Suppose the system Hamiltonian is given in (31), $\bar{F}_i$ is given in (33) and $\bar{M}$ is connected. For a vertex $O_a \in G_{\lfloor k}$, if there exist $O_c, O_d \in G^1_{\lfloor k+1}$ which are adjacent to O_a, we have the following statements.

1) O_c and O_d are $(k+1)$-finite.
2) O_c and O_d are connected.

Lemma 5: Suppose the system Hamiltonian is given in (31), $\bar{F}_i$ is given in (33) and $\bar{M}$ is connected. Two vertices $O_a, O_b \in G_{\lfloor k}$ are adjacent. If there exists $O_c \in G^1_{\lfloor k+1}$ which is adjacent to O_a, we have the following statements.

1) There exists $O_d \in G^1_{\lfloor k+1}$ which is adjacent to O_b.
2) O_c and O_d are $(k+1)$-finite.
3) There exists a path in $G_{\lfloor k+1}$ that connects O_c and O_d.

The main difficulty is in proving Assertions 2 and 3. Define $\bar{F}_{\lfloor k} = \bar{F}_k - \bar{F}_{k-1}$. For the system with Hamiltonian given in (33), we have $\bar{F}_{\lfloor k+1} = \{X_k X_{k+1}, Y_k Y_{k+1}\}$. Since operators in $\bar{F}_{\lfloor k+1}$ can only relate operators in $G_{\lfloor k}$ and $G^1_{\lfloor k+1}$, all of the operators in $G^1_{\lfloor k+1}$ are generated by operators in $G_{\lfloor k}$. Thus, to prove Assertion 2 and Assertion 3, it suffices to consider elements in $G_{\lfloor k}$. We assert that every element in $G_{\lfloor k}$ can generate at most two elements in $G^1_{\lfloor k+1}$ for a system whose Hamiltonian takes the form of (31). For the case where an operator in $G_{\lfloor k}$ can generate two operators in $G^1_{\lfloor k+1}$, we present Lemma 4 to confirm that the generated operators in $G^1_{\lfloor k+1}$ are connected and $(k+1)$-finite. For the case where an operator in $G_{\lfloor k}$ only generates one operator in $G^1_{\lfloor k+1}$, we present Lemma 5 to confirm that the generated operators in $G^1_{\lfloor k+1}$ are connected and $(k+1)$-finite.

Lemma 6: Given $O_1 \in \Omega$ and $\nu_1, \nu_2, \ldots, \nu_m \in \bar{F}$ as in (33) where m is any positive integer. Define two edging sequences

$$\begin{aligned} E_1 &= (\nu_{\epsilon(1)}, \nu_{\epsilon(2)}, \ldots, \nu_{\epsilon(m)}) \\ E_2 &= (\nu_{\epsilon'(1)}, \nu_{\epsilon'(2)}, \ldots, \nu_{\epsilon'(m)}) \end{aligned} \tag{57}$$

where $\epsilon(\cdot)$ and $\epsilon'(\cdot)$ are two choices of the bijective map $\mathcal{F} \triangleq \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$. For the two paths $\{O_1, E_1, O_2\}$ and $\{O_1, E_2, O_3\}$, we have $O_2 = O_3$ if $O_2, O_3 \in \Omega$.

Lemma 7: Let the edging sequence connecting $O_a \in \Omega$ and $O_b \in \Omega$ be denoted as E and assume that $E \in S(\bar{F}_{k+1})$ where $\bar{F}_{k+1}$ is given in (33). For an edging sequence E' such that $C(E') = C(E) - C(E_p)$ where $C(E_p) \subset C(E)$ and each element in $E_p \in S(\bar{F}_{k+1})$ appears an even number of times, if there exists a path $\{O_a, E,' O_c\}$ and $O_c \in \Omega$, then we have $O_a = O_c$.

Here, the subtraction $A - B$ for two collections A and B is defined as removing same operators in B from A. If an operator appears $k (k \geq 2)$ times in B, k such operators should be removed from A.

Lemma 8: Given $\bar{F}_{k+1}$ as in (33), $\bar{M}$ is a decomposed measurement set and the graph is $\mathbb{G} = \{G, \mathbb{E}\}$ where $G = f(\Omega, \bar{F}_{k+1}, \bar{M})$. If there exists a path $\{O_a, E, O_b\}$ where $O_a \in G_k$, $O_b \in G_{k+1}$ and $E = (\nu_1, \nu_2, \cdots) \in S(\bar{F}_{k+1})$, and O_b is not (k+1)-finite, we have $O_b \in G_k$.

We now move to the proof of Proposition 3 using the previous lemmas.

Proof: The proof of Assertion 1 is straightforward. For any $1 \leq l \leq \mu$, given that $G_l = f(\Omega, \bar{F}_l, \bar{M})$ and $G_\mu = f(\Omega, \bar{F}_\mu, \bar{M})$, since $\bar{F}_{\lfloor l} \subseteq \bar{F}_{\lfloor \mu}$, we have $G_l \subseteq G_\mu$. ■

Then we prove Assertion 2 and Assertion 3 at the same time. Essentially, Assertion 2 and Assertion 3 together state that $G_{\lfloor k}$ is k-finite, and, moreover, the graph generated by $G_{\lfloor k}$ is connected.

We use the induction method to prove Assertion 2 and Assertion 3. Suppose that $G_{\lfloor i}$ is i-finite and is connected for any $1 \leq i \leq k$; we prove that $G_{\lfloor k+1}$ is $(k+1)$-finite and $G_{\lfloor k+1}$ is connected. According to Lemma 2, G_{k+1} is connected. Since $G_{k+1} = G_k + G_{\lfloor k+1}$ and $G^2_{\lfloor k+1} \subset G_{\lfloor k+1}$ is not adjacent to G_k, $G^2_{\lfloor k+1}$ must be connected to $G^1_{\lfloor k+1}$. Thus, to prove Assertion 3, it suffices to prove that both the induced subgraphs $\mathbb{G}^1_{\lfloor k+1}$ and $\mathbb{G}^2_{\lfloor k+1}$ are connected given that $\mathbb{G}_k$ is connected. To prove Assertion 2, we need to prove that operators in $G^1_{\lfloor k+1}$ and $G^2_{\lfloor k+1}$ are $(k+1)$-finite given that G_k is k-finite.

Lemma 4 concerns the case where one element operator in $G_{\lfloor k}$ generates two element operators in $G^1_{\lfloor k+1}$ and Lemma 5 concerns the case where one element operator in $G_{\lfloor k}$ generates only one element operator in $G^1_{\lfloor k+1}$. For both cases, the generated operators are connected and $(k+1)$-finite. Then, from Lemma 4, Lemma 5, and the assumption that Assertions 2 and 3 hold for $G^1_{\lfloor k}$, Assertions 2 and 3 hold for $G^1_{\lfloor k+1}$. Since $G^2_{\lfloor k+1}$ can be generated by $G^1_{\lfloor k+1}$, the set $G_{\lfloor k+1}$ is connected, which means Assertion 3 holds for $G_{\lfloor k+1}$.

Having proved that Assertion 2 holds for $G^1_{\lfloor k+1}$, we prove that Assertion 2 also holds for $G^2_{\lfloor k+1}$. For $O_b \in G_{\lfloor k+1}$, there must be a path $\{O_a, E, O_b\}$ where $O_a \in G_k$ and all of the vertices in the path except O_b are in $G_{\lfloor k} \cap G_{\lfloor k+1}$ because we have the assumption that G_k is connected and previously proved that G^1_{k+1} is connected. Then from Lemma 8, if O_b is not $(k+1)$-finite, we have $O_b \in G_k$, which contradicts the assumption that $O_b \in G_{\lfloor k+1}$. Then we conclude that the set $G^2_{\lfloor k+1}$ is $(k+1)$-finite. Thus Assertion 2 is proved.

So far, we have proved that $G_{\lfloor k+1}$ is connected and the set $G_{\lfloor k+1}$ is $(k+1)$-finite, given that Assertion 2 and Assertion 3 apply to G_k. In our case, we assume that $\mathbb{G}_1 = \mathbb{M}$ is connected. Thus, we can always find a k validating Assertion 2 and Assertion 3. Therefore, using the induction method, Proposition 3 is proved.

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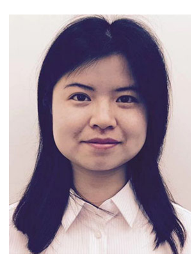

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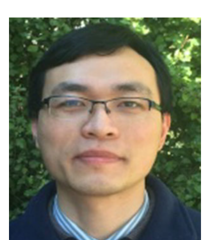

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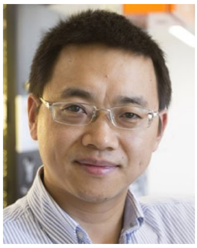

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