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Ratio-based data envelopment analysis: An interactive approach to identify benchmark

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ABSTRACT

In the real world we are confronted with many cases where the ratio of input/output data is so important for managers, so in this regard we cannot use traditional Data Envelopment Analysis (DEA) models to evaluate the efficiency of decision-making units (DMU) and we should use DEA models based on ratio data. To get the corresponding benchmark for each inefficient decision-making unit, we need to reduce and increase the inputs and outputs, respectively, and get the unified projection of the decision maker unit on the efficiency frontier. In this paper we present a multi-objective linear programming (MOLP) model for evaluating efficiency based on defining the production possibility set in the presence of ratio data and to get the corresponding benchmark to each decision-making unit DMU. We use the Zionts and Wallenius (Z–W) interactive method to solve the MOLP model presented. Using the target setting by manager among the solutions resulting from the MOLP problem, we choose best solution according to the managers' preferences as benchmark and at the end we present the results of the research.

1. Introduction

Farrell [1] was the first researcher to make an evaluation of decision-making units (DMUs) based on two inputs and one output using non-parametric methods. The system proposed analyzes the performance of units. Using mathematical programming, Charnes et al. [2] generalized Farrell's non-parametric method for multiple inputs and outputs and introduced it as the CCR model since it was developed by Charnes, Cooper, and Rhodes where benchmarks are calculated for the DMUs based on the optimal model solutions. This means that if the optimal values are obtained in the radial and non-radial models CCR or BCC [3], the benchmark of inefficient units can then be easily achieved by placing constraints on inputs and outputs. The benchmarks of an inefficient unit can be real or virtual and these benchmarks can be used to make the final decision according to the manager's opinion. In cases where we have ratio data such as the ratio of staff to students at a university or the ratio of capital to debt in a financial institution, it is obvious that traditional DEA models can no longer evaluate our units and find the benchmarks. DEA models need the inputs and outputs of a DMU for proper benchmarking. Because of this, several studies have been conducted in order to identify the DEA benchmarks.

A virtual reference unit is determined for each inefficient unit by the available efficient units. The set of these units is called the reference set. One definition of efficiency in this model is the ratio of the weighted sum of outputs to the weighted sum of inputs. In this model, DMUs are evaluated in their best conditions, i.e. the best prices are assigned to input and output weights. What is

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important in DEA models is the use of non-Archimedean numbers for weights, which would guarantee positive input and output weights. This may cause false inefficiency since the efficiency value does not show the actual efficiency of the units evaluated. Moreover, when our data are in ratios, DEA models cannot provide an appropriate response. To solve this problem, Despic et al. [4] introduced the DEA-R model for the first time by combining the DEA methodology and Ratio Analysis. In solving linear programming problems through interactive methods, there is no need to obtain the decision-maker's preference information before solving the model. Meanwhile, the information is gathered during the solving process. In other words, it is not necessary for the decision maker (DM) to state his/her overall priorities in relation to the objectives, but rather they just simply indicate their preferences from a set of available solutions, which is easier for the DM. Developed by Zionts and Wallenius [5], the Z–W method is an interactive multi-objective linear programming approach used to obtain a proper benchmark for inefficient units on the efficient frontier. In this method, an additive and combined objective function is first formed using positive and arbitrary weights, and then the effective solution is achieved by solving the problem.

DEA-R utilizes linear programming models to evaluate DMUs and find appropriate benchmarks. In cases of ratio data, the scale efficiency is calculated by defining efficiency as a weighted sum of input-to-output ratios or vice versa. In this regard, the role of the manager is uniquely important. However, it is obvious that the use of interactive methods, especially the Z–W method, is essential for benchmarking in DEA-R since, firstly, data are in form of ratios, and secondly, by incorporating the manager's preferences, we can achieve the proper benchmarks and ensure the benchmarks are in line with the manager's ideas. In this paper, input and output data are not inherently ratios, but these ratios form the criteria for assessing DMUs.

The axiom of convexity is one of the basic assumptions of the production possibility set (PPS). Emrouznejad and Amin [6] addressed the subject of convexity in a standard PPS. In this regard, they stated that using the basic models of standard DEA for observations including ratio data as inputs and outputs could lead to inaccurate scale efficiencies. Therefore, they presented modified DEA models that employ an accurate convexity of DMUs in cases where the model includes ratio variables. Olesen et al. [7] demonstrated that using ratio data (e.g., different percentages or rates) in variable returns to scale (VRS) and constant returns to scale (CRS) models would contradict the production assumptions. If ratio measures are used, the production technology would become non-convex. Thus, standard DEA models are no longer suitable when one of the inputs or outputs is in the form of a ratio. In another study, Olesen et al. [8] introduced the new R-CRS and R-VRS production technologies, which can be considered as extensions of standard DEA models that allow the incorporation of ratio inputs and outputs.

The discussion about DEA-R is quite different from other performance measurement studies using ratio data. DEA-R is strictly modeled based on ratio analysis, where the inputs and outputs of DMUs are available in a non-ratio form, but their ratios can be computed and form the criteria for evaluating the DMUs. Ratios such as quick ratio and leverage ratio are examples of criteria that could be used to evaluate companies, should the production technology be defined accordingly, as in the studies of Fernandez-Castro and Smith [9], Liu et al. [10], Mozaffari et al. [11].

The present paper is structured as follows. Section 2 presents literature review in different three parts, including relation between DEA and multi-objective linear programming (MOLP), benchmarking, and DEA-R models. Section 3 presents the basic concepts in DEA-R, or DEA with ratio data. In Section 4, we propose a DEA-R model with a MOLP structure used for benchmarking and provide a demonstration of the interactive Z–W method along with a stepwise implementation of the model. The algorithm proposed is illustrated through numerical and applicatory examples in Section 5. In Section 6 we benchmark eleven Iranian clothing companies using DEA and DEA-R models, while finally in Section 7 the conclusive remarks are made.

2. Literature review

We proposed the literature review in different three parts. The first part is a review of relation of MOLP and DEA. The second part is a review benchmarking in DEA. We also review the recent DEA-R models in the thread part.

2.1. MOLP and DEA

The conventional DEA models do not incorporate the DM's preference or value judgments. Different methods have been developed to consider the DM's preference information in the efficiency evaluation process. Allen et al. [12] have defined value judgments as "logical constructs, in order to incorporate DM's preference information in the process of assessing efficiency. Golany [13] proposed the goal and target setting models. Athanassopoulis [14], and Dyson and Thanassoulis [15] developed weight restrictions models including imposing bounds on individual weights. Thompson et al. [16] proposed assurance region method. Wong and Beasley [17] developed restricting composite inputs and outputs, weight ratios and proportions based on the DEA models. Charnes and Cooper [18], and Charnes et al. [19] used the cone ratio concept by adjusting the observed input–output levels or weights to apply value judgment to belong to a given closed cone. Zhu [20] integrated preference information into a modified DEA formulation. But, all the above-mentioned methods would require prior articulated preference knowledge from the DM, which in most cases can be subjective and difficult to obtain. An another method to consider preference information, without necessary prior judgment or target setting, is the use of an interactive decision making technique that encompasses both DEA and Linear Programming (MOLP). Joro, Korhonen, and Wallenius [21] showed that there are synergies from both DEA and MOLP, and showed that the DEA formulation is structurally similar to the reference point approach of the MOLP formulation. They used of the concept of value efficiency analysis to incorporate preference information in DEA. Joro, Korhonen and Zionts [22] proposed an interactive approach to improve estimates of value efficiency in data envelopment analysis. Wong, Luque, and Yang [23] used interactive multi objective methods to solve DEA problems with value judgments. Yang, Wong, Xu, and Stewart [24] proposed an integrating

DEA-oriented model for assessment performance and target setting using interactive MOLP methods. Hosseinzadeh Lotfi et al. [25] developed a relationship between MOLP and DEA based on output-orientated CCR dual model. Hosseinzadeh Lotfi et al. [26] proposed target setting in the general combined-oriented CCR model using an interactive MOLP method. Zohrehbandian [27] used Zionts–Wallenius method to improve estimate of value efficiency in DEA. Ebrahimnejad and Hosseinzadeh Lotfi [28] developed an equivalence relationship between the general combined-oriented CCR model and the weighted minimax MOLP formulation. Ebrahimnejad and Tavana [29] proposed an interactive MOLP method for identifying target units in output-oriented DEA models. Ebrahimnejad et al. [30] proposed a three-stage data envelopment analysis model with application to banking industry. Kao, Chan, and Wu [31] proposed a multi-objective programming method for solving network DEA. Gerami [32] proposed an interactive procedure to improve estimate of value efficiency in DEA. Soltani and Lozano [33] proposed an interactive multi objective DEA target setting using lexicographic DDF.

2.2. Benchmarking in DEA

Benchmarking widespread used to improve a firm's performance relative of DMUs in the DEA. DEA employed as a practical benchmarking tool in management. The procedure of DEA-based benchmarking approaches can be concluded in the following four steps.

1. The input/output dimensions and indexes are obtained. 2. the relevant data set of all the evaluated DMUs are considered, and a DEA model is selected to discriminate DMUs with superior performance and other DMUs with inferior performance. 3. By considering a specific criterion, the inefficient DMUs can find rational benchmarks among the best performing DMUs or their linear combinations called virtual efficient DMUs. 4. The inefficient DMUs improve their performance toward those benchmarks set by DEA method. The study on benchmarking is a main research field in DEA. For example, see: Cook, Ruiz, Sirvent, and Zhu [34], Lim, Bae, and Lee [35], Lozano, Calzada-Infante [36].

Lim, Bae, and Lee [35] proposed a study on the selection of benchmarking paths in DEA. Cook, Tone, and Zhu [37] proposed an approach based on DEA in the prior to choosing a model. Bogetoft [38] proposed the original definition of benchmarking. He showed that benchmarking is traditionally thought of as a managerial tool that improves performance by obtaining and applying best documented practices in the business world, he indicated that benchmarking incorporates the identification of best-performance goals and the application of these goals into practice. Cook, Ruiz, Sirvent, and Zhu [34] proposed within-group common benchmarking using DEA. Lozano and Soltani [39] proposed DEA target setting using lexicographic and endogenous directional distance function approaches. An, Tao, and Xiong [40] proposed an agency perspective of benchmarking with data envelopment analysis. They propose a reimbursement scheme to motivate DMUs to realize their "best practice" and prove that DMUs' best responses to our incentive game are just to realize their "best practice" when the reimbursement scheme satisfies strong monotonicity in outputs, and these responses constitute the strong Nash equilibrium of our incentive game. José and Sirvent [41] presented common benchmarking and ranking of units with DEA. Soltani and Lozano [33] proposed an interactive multi objective DEA target setting using lexicographic DDF.

2.3. DEA-R model

DEA-R models were first formulated in Despic et al. [4] as a tool that combines DEA and ratio analysis. Emrouzinejad and Amin [6] proposed a new convexity assumption as well as enhancements to basic DEA models to tackle this problem. Emrouznejad and Amin [6] showed that the convexity assumption is not satisfied. Wei et al. [42–44] extended the theory of DEA-R models in new directions. They presented on relations between traditional DEA models and ratio-based DEA-R. Liu et al. [10] proposed DEA-R models without explicit inputs studied. Mozaffari et al. [45] developed DEA-R models based on the cost and revenue efficiency concept and the relationship between DEA and DEA-R. Olesen et al. [7,8], having demonstrated the problems with ratio data after classifying them, defined a production possibility set and introduced the corresponding models in constant/variable returns to scale technology. They discussed efficiency analysis with ratio measures and provided a positive answer to the existing debate with regard to the use of DEA models for ratio data. They proposed a new production possibility set under VRS and CRS production technologies. Mozaffari et al. [11] introduce a DEA-R production possibility set under the assumption of constant returns to scale technology and propose a method for identifying DEA-R-efficient surfaces. Hatami-Marbini and Toloo [46] shown the problems with ratio data after classifying them, defined a production possibility set and introduced the corresponding models in constant/variable returns to scale technology. Gerami et al. [47] proposed multi-criteria ratios for two-stage network. Finally, Gerami et al. [48] developed a novel network DEA-R model for evaluating hospital services supply chain performance. The DEA models have the properties: in the DEA-R model, the inputs and outputs are available and are not ratio data. In DEA-R models, we use ratios of inputs to outputs or vice versa. In contrast to DEA, efficiency is defined in DEA-R as a weighted sum of input-to-output ratios or vice versa, a definition of efficiency based on the relationship between arithmetic, geometric, and harmonic efficiency (Despic et al. [4]). In DEA-R models, pseudo-inefficiency is prevented, and the efficiency scores and the DEA and DEA-R weights have a similar behavior. (Wei et al. [42–44]).

3. Preliminaries

In this section, we first provide our defined PPS and the proposed output-oriented envelopment models in DEA-R, then we present our ratio-based DEA models adopted from articles by Emrouznejad and Amin [6] and Olesen et al. [8].

3.1. An overview of basic concepts in DEA- R

Consider n decision-making units (DMUs) that use m inputs to produce s outputs. Let us assume that we have n DMUs (DMUj, $j = 1, \ldots, n$), each associated with m inputs $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^T$ and s outputs $Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^T, j = 1, \ldots, n$. The production possibility set (PPS) is then introduced as follows.

$T = \{(y, x) \,|\, y \text{ can be produced from } x\} \in R_+^{m+p}$.

The following shows the output-oriented CCR envelopment model used to assess the efficiency of DMUs.

$$\begin{aligned} Max \quad & \varphi \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}, \qquad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge \varphi y_{ro}, \qquad r = 1, \ldots, s, \\ & \lambda_j \ge 0, \qquad j = 1, \ldots, n. \end{aligned} \tag{3.1}$$

In the optimal solution of model (3.1), if $\varphi^* = 1$, DMUo is located on the efficiency frontier and is thus efficient. Next, we consider the model's second phase using the θ^* value and maximizing the set of slack variables related to input and output constraints. Now, if $(\lambda_1, \ldots, \lambda_n, \theta^*)$ was the optimal solution in the second phase of model (3.1), then $(\sum_{j=1}^{n} \lambda_j^* x_{ij}, \sum_{j=1}^{n} \lambda_j^* y_{rj})$ would be the efficiency projection of DMU_O, thereby separating strong and weak efficiencies [49]. In addition, this model is always feasible as $\theta^* = 1$, $\lambda_o = 1$, and $\lambda_j = 0, j = 1, \ldots, n, j \ne o$, is a feasible solution. Furthermore, in any feasible solution, especially the optimal solution, we have $\varphi^* \geq 1$.

We define the production possibility set (PPS) in the presence of ratio data as follows:

$$T_R = \left\{ \frac{y}{x} \,\middle|\, \sum_{j=1}^{n} \lambda_j \left(\frac{y_{rj}}{x_{ij}} \right) \ge \frac{y}{x} \ r = 1, \ldots, s, \ i = 1, \ldots, m \ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge 0, j = 1, \ldots, n \right\} \tag{3.2}$$

The following postulates hold in the T_R set (3.2):

- **Feasibility of data observed**: The inclusion principle of observations related to the ratios of $\frac{y_{rj}}{x_{ij}}$ holds; $\left(\forall j = 1, \ldots, n, \quad \frac{y_{rj}}{x_{ij}} \in T_R \right)$.
- **Free disposability:** This principle holds in the DEA-R production possibility set (see [10]); $\left(if \ \frac{\beta}{\alpha} \in T_R \ \forall \frac{y}{x}, \quad \frac{y}{x} \le \frac{\beta}{\alpha} \ \rightarrow \frac{y}{x} \in T_R, \ j = 1, \ldots, n \right)$.
- **Convexity:** The convexity principle holds in the DEA-R PPS (see [10]); (If $\left(\frac{y^1}{x^1}\right), \left(\frac{y^2}{x^2}\right) \in T_R$ and $\lambda \in [0, 1] \rightarrow \lambda \left(\frac{y^1}{x^1}\right) + (1 - \lambda) \left(\frac{y^2}{x^2}\right) \in T_R$).

Theorem 1. *T_R is a closed and bounded set (see [10]).*

According to the above mentioned principles, in the presence of ratio data, we define the output-oriented DEA-R model under constant returns to scale (CRS) assumption as follows:

$$\begin{aligned} Max \quad & \varphi \\ s.t. \quad & \varphi \left(\frac{y_{ro}}{x_{io}} \right) \in T_R \quad i = 1, \ldots, m, \quad r = 1, \ldots, s \end{aligned} \tag{3.3}$$

Based on the PPS defined, the above mentioned model will be presented as follows:

$$\begin{aligned} Max \quad & \varphi \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j \left(\frac{y_{rj}}{x_{ij}} \right) \ge \varphi \left(\frac{y_{ro}}{x_{io}} \right), \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \ge 0, \quad j = 1, \ldots, n. \end{aligned} \tag{3.4}$$

Model (3.4) is a linear programming problem introduced in DEA-R to assess technologies with CRS [42–44]. In DEA models, efficiency is equal to the weighted sum of outputs divided by the weighted sum of inputs, and relative efficiency is defined as the absolute efficiency divided by maximum absolute efficiencies. The following problems exist in this relation: First, what are our reasons for defining efficiency? Secondly, use of the non-Archimedean number (ε) is an issue as it prevents zero weights such that neither the nominator nor the denominator can become zero. Thirdly, the aforementioned problems may lead to the pseudo-inefficiency in DEA. Here, the DEA-R model is useful for our purposes and it does not cause any problems. In addition, the efficiency score in input-oriented models of DEA-R is greater than or equal to the value efficiency score in DEA. Efficiency scores are the exact equivalent of each other in DEA and DEA-R models when there is one output and multiple inputs, which can be easily proved (see [44]).

3.2. Data envelopment analysis with ratio data

Fernandez-Castro and Smith [9] provide the following model for calculating the efficiency of unit under evaluation, ie $DMU_o = \left(x_o, y_o\right)$ as follows.

$$\begin{aligned} \max \quad & \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ir}\left(\frac{y_{ro}}{x_{io}}\right) \\ s.t. \quad & \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ir}\left(\frac{y_{rj}}{x_{ij}}\right) \le 1, \quad j=1,\dots,n, \\ & w_{ir} \ge 0, \quad i=1,\dots,m, \quad r=1,\dots,s, \end{aligned} \tag{3.5}$$

This model has been used in Meng et al. [50]. Model (3.5) also looks like a DEA model without inputs; see Mahlberg and Obersteiner [51].

If we write the dual of model (3.5), we get to model (3.6).

$$\begin{aligned} Min \quad & \sum_{j=1}^{n} \lambda_j \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j\left(\frac{y_{rj}}{x_{ij}}\right) \ge \left(\frac{y_{ro}}{x_{io}}\right), \ i=1,\dots,m, \ r=1,\dots,s, \\ & \lambda_j \ge 0, \quad j=1,\dots,n. \end{aligned} \tag{3.6}$$

By changing the variables $\hat{\lambda}_j = \frac{\lambda_j}{t}$ and $t = \sum_{j=1}^{n}\lambda_j$ and $\hat{\theta}_R = \frac{1}{t}$ in model (3.6) and converting the maximization problem to a minimization problem, the model (3.7) is obtained as follows.

$$\begin{aligned} Max \quad & \hat{\theta}_R \\ s.t. \quad & \sum_{j=1}^{n} \hat{\lambda}_j\left(\frac{y_{rj}}{x_{ij}}\right) \ge \hat{\theta}_R(\frac{y_{ro}}{x_{io}}), \quad i=1,\dots,m, \quad r=1,\dots,s, \\ & \sum_{j=1}^{n} \hat{\lambda}_j = 1, \quad \hat{\lambda}_j \ge 0, \quad j=1,\dots,n, \quad \hat{\theta}_R \ge 0. \end{aligned} \tag{3.7}$$

Model (3.7) is the same model DEA-R-O (3.4), which was introduced in Section 3.1. As observed, we obtained the DEA-R-O model in two different ways to calculate the efficiency of the unit under evaluation.

4. Benchmarking in DEA-R

In this section we first propose a multi-objective linear programming problem based on DEA-R in constant returns to scale (CRS) technology, then use the interactive Z–W method to obtain benchmarks for our DMUs on the efficiency frontier.

By defining the PPS in DEA-R, i.e. model (3.4), the following multi-objective model is proposed:

$$\begin{aligned} Max \quad & \left\{f_{ir}, \quad i=1,\dots,m, \quad r=1,\dots,s\right\} \\ S.t. \quad & \sum_{j=1}^{n} \lambda_j\left(\frac{y_{rj}}{x_{ij}}\right) \ge f_{ir}, \quad i=1,\dots,m, \quad r=1,\dots,s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \ge 0, \quad j=1,\dots,n. \end{aligned} \tag{4.8}$$

Theorem 2. *The Pareto-optimal solutions of model (4.8) are on the output-oriented DEA-R efficiency frontier.*

Proof. Model (4.8) is an $m \times s$ objective linear programming problem. We prove that the Pareto-optimal solutions of this model are on the output-oriented DEA-R efficiency frontier. To prove our theorem by contraction, assume that $\left(\lambda_j^*, f_{ir}^*\right)$ is a Pareto-optimal solution for model (4.8) that is not on the DEA-R efficient frontier. Therefore, we let $f_{ir}^* = \frac{\tilde{\beta}_r}{\tilde{\alpha}_i}$ that is not on the efficiency frontier be our solution.

Since $f_{ir}^* = \frac{\tilde{\beta}_r}{\tilde{\alpha}_i}$ is not on the frontier, then $\exists \hat{\lambda}_j \ni \sum_{j=1}^{n}\hat{\lambda}_j\frac{y_{rj}}{x_{ij}} \ge \frac{\tilde{\beta}_r}{\tilde{\alpha}_i}$ assuming $\sum_{j=1}^{n}\hat{\lambda}_j\frac{y_{rj}}{x_{ij}} = \frac{\hat{\beta}_r}{\hat{\alpha}_i}$ is thereby $\left(\hat{\lambda}_j, \frac{\hat{\beta}_r}{\hat{\alpha}_i}\right)$ a feasible solution for model (4.8), which $\hat{f}_{ir} = \frac{\hat{\beta}_r}{\hat{\alpha}_i}$ is in contraction with the Pareto-optimal target value of model (4.8). Therefore, the assumption is false and our statement holds.

We used the weighted sum method to solve our multi-objective problems. Assigning weights to γ_{ir}, which represent the priorities of output to input ratios, model (4.9) is suggested as follows:

$$\begin{aligned}
Max \quad & \sum_{i=1}^{m}\sum_{r=1}^{s} f_{ir}\gamma_{ir} \\
S.t. \quad & \sum_{j=1}^{n}\lambda_j\left(\frac{y_{rj}}{x_{ij}}\right) \geq f_{ir}, \quad i=1,\dots,m, \quad r=1,\dots,s, \\
& \sum_{j=1}^{n}\lambda_j = 1,\ \lambda_j \geq 0, \quad j=1,\dots,n.
\end{aligned} \tag{4.9}$$

We now obtain the optimal solutions of model (4.9) and its projection on the efficiency frontier as follows:

$$x_i^* = \sum \lambda_j^* x_{ij},\ y_r^* = \sum \lambda_j^* y_{rj},\ \sum_{j=1}^{n}\lambda_j^* = 1, i=1,\dots,m,\ r=1,\dots,s.,$$

Theorem 3. *Model (4.9) is always feasible*

Proof. If we let $\overline{\lambda}_o = 0$ for $j=1,\dots,n, j\neq$ o, as well as $\overline{\lambda}_o = 1$ and $\overline{f}_{ir} = \frac{x_{io}}{y_{ro}}$ for all *i*, *r*, *a* values, the feasible solution for model (4.9) is then obtained, which holds for all constraints.

Theorem 4. *The first constraints in model (4.9) are binding in the optimal solutions*

Proof. Suppose that

$$\left(\lambda_1^*,\dots,\lambda_n^*, f_{11}^*,\dots,f_{1s}^*, f_{21}^*,\dots,f_{2s}^*,\dots,f_{m1}^*,\dots,f_{ms}^*\right)$$

is the optimal solution of model (4.9). By contradiction, suppose there are t and k, for which the corresponding constraints are not binding; i.e.,

$\exists t,k:\ \sum_{j=1}^{n}\lambda_j^*\left(\frac{y_{kj}}{x_{tj}}\right) > f_{tk}^*$

Obviously, there is $\overline{f}_{tk}$, per which:

$$\sum_{j=1}^{n}\lambda_j^*\left(\frac{y_{kj}}{x_{tj}}\right) = \overline{f}_{tk} \& f_{tk}^* < \overline{f}_{tk},$$

because $\overline{f}_{tk} = \sum_{j=1}^{n}\lambda_j^*\left(\frac{y_{kj}}{x_{tj}}\right) > f_{tk}^*$. Now, we define:

$$\overline{f}_{ir} = f_{ir}^* \quad \begin{matrix} i=1,\dots,m, & i\neq t \\ r=1,\dots,s, & r\neq k \end{matrix}$$

It is obvious that:

$$\left(\lambda_1^*,\dots,\lambda_n^*,\overline{f}_{11},\dots,\overline{f}_{1s},\overline{f}_{21},\dots,\overline{f}_{2s},\dots,\overline{f}_{m1},\dots,\overline{f}_{ms}\right)$$

is a feasible solution for model (4.9). Therefore, using the weight vector used to solve the problem, we will arrive at the following equation, which is a contradiction:

$$\sum_{i=1}^{m}\sum_{r=1}^{s}\gamma_{ir}\overline{f}_{ir} > \sum_{i=1}^{m}\sum_{r=1}^{s}\gamma_{ir}f_{ir}^*$$

Therefore, this completes the proof.

4.1. Benchmarking in DEA-R via interactive methods

The first study to combine DEA and MOLP was conducted by Golany [13]. He suggested using an interactive method for generating efficient solutions in order to determine the efficient frontier in DEA, which was somewhat similar to the weighted Tchebychev method presented by Steuer and Choo [52]. We propose the following algorithm for finding the benchmarks in DEA-R.

Step 1. First, we write the objective function as Max $\{f_{ir}\}$ and select a set of weights γ^h for the objectives of f_{ir}, then we let $h=1$.

Step 2. We create an objective function for the problem, as follows, and solve it in order to achieve an effective solution.

$$\begin{aligned}
Max \quad & \sum_{i=1}^{m}\sum_{r=1}^{s}\gamma_{ir}^h f_{ir} \\
S.t. \quad &
\end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j \left(\frac{y_{rj}}{x_{ij}}\right) \geq f_{ir}, \quad i = 1, \dots, m, \quad r = 1, \dots, s,$$
$$\sum_{j=1}^{n} \lambda_j = 1 \;, \lambda_j \geq 0, \qquad j = 1, \dots, n, \tag{4.10}$$
$$\sum_{i=1}^{m} \sum_{r=1}^{s} \gamma_{ir}^{h} = 1, \gamma_{ir}^{h} > 0, i = 1, \dots, m, r = 1, \dots, s.$$

Step 3. We gather the optimal solutions of model (4.10) in the set F^1 and assign the non-basic variable. Solve the model (4.10) by considering the positive weights $\gamma_{ir}^{h} > 0$. Because the model (4.10) is a linear programming problem and each optimal solution of it is an optimal Pareto solution of the multi-objective model (4.8) (Steuer and Choo [52]).

So assuming that $\left(\lambda_1^*, \dots, \lambda_n^*, f_{11}^*, \dots, f_{1s}^*, f_{21}^*, \dots, f_{2s}^*, \dots, f_{m1}^*, \dots, f_{ms}^*\right)$ is an optimal solution of the linear programming model (4.10). Obviously, any optimal solution of the model (4.10) can be divided into basic and non-basic solution. It should be noted that in the non-degeneracy state, all the basic variables are positive and the other non-basic variables are at the zero level. Therefore, at this stage, only non-basic variables are considered. Variables (NBVs) to the set q_1; furthermore, we set the index of non-basic variables to NBV.

Step 4. In order to calculate ω_{kl}, $k = 1, \dots, m \times s$ for the entry of non-basic variable q_l into the base, the following problem should be solved for the non-basic variables q_l. Solving the model (4.10) and separating the basic and non-basic variables, and then solve the models (4.11a), (4.11b) for the basic variables. Simplicity, we can first find the non-basic variables among the non-basic variables in between

$\left(\lambda_1^*, \dots, \lambda_n^*\right)$ and solve the following model for $j \in NBV$ that NBV is a set of all non-basic variables.

$$\begin{aligned} Max \quad & \{\lambda_j : j \in NBV\} \\ S.t. \quad & \sum_{j=1}^{n} \lambda_j \left(\frac{y_{rj}}{x_{ij}}\right) \geq f_{ir}, i = 1, \dots, m, \; r = 1, \dots, s, \\ & \sum_{J=1}^{n} \lambda_j = 1 \;, \lambda_j \geq 0, \; j = 1, \dots, n. \end{aligned} \tag{4.11a}$$

Similarly, we can select non-basic variables from among the non-basic variables $\left(f_{11}^*, \dots, f_{1s}^*, f_{21}^*, \dots, f_{2s}^*, \dots, f_{m1}^*, \dots, f_{ms}^*\right)$, and we solve the following model for $(i,r) \in NBV$, where NBV is a set of all non-basic variables.

$$\begin{aligned} Max \quad & \{f_{ir} : ir \in NBV\} \\ S.t. \quad & \sum_{j=1}^{n} \lambda_j \left(\frac{y_{rj}}{x_{ij}}\right) \geq f_{ir}, i = 1, \dots, m, \; r = 1, \dots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \quad j = 1, \dots, n. \end{aligned} \tag{4.11b}$$

Considering that the models (4.11b), (4.11a) may have the alternative optimal solution, so in this case we can find the variable corresponding to the optimal solution obtained from the models (4.11a), (4.11b) which has a larger value. Because only the maximum of that non-basic variable is important.

We suppose that the optimal solutions of model (4.11a) or (4.11b) are obtained as follows:

$$F'^1 = \left(f_{11}'^1, f_{12}'^1, \dots, f_{1s}'^1, f_{21}'^1, f_{22}'^1, \dots, f_{2s}'^1, \dots, f_{m1}'^1, \dots, f_{ms}'^1\right)$$
$$\vdots$$
$$F'^N = \left(f_{11}'^N, f_{12}'^N, \dots, f_{1s}'^N, f_{21}'^N, f_{22}'^N, \dots, f_{2s}'^N, \dots, f_{m1}'^N, \dots, f_{ms}'^N\right).$$

So if we consider the index corresponding to the non-basic variable that the maximum corresponding to the model (4.11a) as $q_l = \lambda_{j_1}^*$, $j_1 \in NBM$, in this case, the model (4.11a) has a larger objective function value than the model (4.11b). We show

$$\omega^l = \left(\omega_{kl}\right)_{m \times s} = \left(\omega_{1l}, \dots, \omega_{m \times sl}\right) = \frac{F^1 - F'^1}{\lambda_{j_1}^*}, \tag{4.12a}$$

Similarly, we consider the index corresponding to the non-basic variable that the maximum corresponding to the model (4.11b) as $f_{i_{1r_1}}^*$, $q_l = \left(i_1, r_1\right) \in NBM$.In this case, the model (4.11b) has a larger objective function value than the model (4.11a). We show

$$\omega^l = \left(\omega_{kl}\right)_{m \times s} = \left(\omega_{1l}, \dots, \omega_{m \times sl}\right) = \frac{F^1 - F'^1}{f_{i_{1r_1}}^*}, \tag{4.12b}$$

Step 5. Now we solve the linear program (4.13) for the non-basic variable $q_l, l \in N$.

$$Min \quad \sum_{k=1}^{m \times s} \omega_{kl} \gamma_k$$

$$
\begin{aligned}
S.t. \quad & \sum_{k=1}^{m\times s} \omega_{kj} \gamma_k \geq 0, \quad j \in N, \quad j \neq l, \\
& \sum_{k=1}^{m\times s} \gamma_k = 1, \\
& \gamma_k \geq 0, \quad k = 1, \dots, m \times s.
\end{aligned} \tag{4.13}
$$

Step 6. When solving the problem in step 4, if the minimum objective function becomes negative, $q_l, l \in N$ is an effective variable, and if this minimum objective function becomes non-negative, the q_l variable will not be effective. The variable is ineffective if every ω_{kl}, $k = 1, \dots, m \times s$ is positive.

Step 7. For every affecting variable $q_l, l \in N$, the DM is asked whether he/she consents to the available alternatives of ω_{kl}, $k = 1, \dots, m \times s$ among the objectives.

(i) If all solutions are "No" for the affecting variables, we stop going further and vector γ^h determines the most effective weight for the optimal function and thus, the problem is assumed to be solved. Otherwise,

(ii) For each "Yes" solution, we form the following inequality:

$$\sum_{k}^{m\times s} \omega_{kl} \gamma_k \leq -\epsilon$$

ϵ is a positive and sufficiently small number. The reason behind the formation of this inequality is that the maximum value of $\sum_{k}^{m\times s} \omega_{kl} \gamma_k$ is always smaller than zero when there is a suitable exchange.

(iii) For each "No" solution, we form the following inequality:

$$\sum_{k}^{m\times s} \omega_{kl} \gamma_k \geq \epsilon$$

iv. For each indifferent solution, we form the following equality:

$$\sum_{k}^{m\times s} \omega_{kl} \gamma_k = \epsilon$$

Step 8. We find a practical solution to the following restrictions:

$$
\begin{aligned}
& \sum_{k}^{m\times s} \omega_{kl} \gamma_k \leq -\epsilon && \text{for a "}Yes\text{" solution} \\
& \sum_{k}^{m\times s} \omega_{kl} \gamma_k \geq \epsilon && \text{for a "}No\text{" solution} \\
& \sum_{k}^{m\times s} \omega_{kl} \gamma_k = \epsilon && \text{for each indifferent solution} \\
& \sum_{k=1}^{m\times s} \gamma_k = 1, && \gamma_k \geq 0, \quad k = 1, \dots, m \times s.
\end{aligned} \tag{4.14}
$$

γ_{ks} obtained from the above mentioned system provides us with new weights in order to solve the existing problem. Now we let $h = h + 1$ and proceed to the second stage.

The following model is suggested for cases where we only have access to a ratio of input to output data:

$$
\begin{aligned}
Min \quad & \left\{ g_{ir}, i = 1, \dots, m, \; r = 1, \dots, s \right\} \\
S.t. \quad & \sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}} \right) \leq g_{ir} \quad i = 1, \dots, m, \quad r = 1, \dots, s, \\
& \sum_{j=1}^{n} \mu_j = 1 \;, \mu_j \geq 0, \; j = 1, \dots, n.
\end{aligned} \tag{4.15}
$$

Model (4.15) is an input-oriented multi-objective programming problem.

4.2. Benchmarking in DEA-R via interactive methods

We define the production possibility set (PPS) in Output-oriented BCC model in DEA-R as follows:

$$
T_{R-VRS} = \left\{ \frac{y}{x} \middle| \sum_{j=1}^{n} \lambda_j \left(\frac{y_{rj}}{x_{ij}} \right) \geq \frac{y}{x}, \sum_{j=1}^{n} \lambda_j \left(\frac{1}{x_{ij}} \right) = \frac{1}{x}, r = 1, \dots, s, \; i = 1, \dots, m, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n \right\} \tag{4.16}
$$

By defining the PPS in DEA-R, i.e. model (4.16), the following multi-objective model is proposed:

$$Max \quad \left\{ Q_{ir}, P_i, i = 1, \dots, m, \quad r = 1, \dots, s \right\}$$

Table 1
The inputs and outputs of four DMUs.

	DMU_1	DMU_2	DMU_3	DMU_4
I_1	2	2	8	2
I_2	3	5	5	4
O	4	4	2	5

$$
\begin{aligned}
S.t.\quad & \sum_{j=1}^{n} \lambda_j \left(\frac{y_{rj}}{x_{ij}}\right) \ge Q_{ir}, \quad i = 1,\dots,m, \quad r = 1,\dots,s,\\
& \sum_{j=1}^{n} \lambda_j \left(\frac{1}{x_{ij}}\right) = P_i, \quad i = 1,\dots,m,\\
& \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \ge 0, \quad j = 1,\dots,n.
\end{aligned} \tag{4.17}
$$

Obviously, model (4.17) is always possible by considering the following vector.
$\overline{\lambda}_j = 1$ for $j = 1,\dots,n$, j ≠ o, as well as $\overline{\lambda}_o = 1$ and $\overline{Q}_{ir} = \frac{y_{ro}}{x_{io}}$ for all i, r, $\overline{P}_i = \frac{1}{x_{io}}$ for all i.
Similarly, DEA-R model in Form BCC based on the MOLP structure is as follows, in which the input to output ratios are defined.

$$
\begin{aligned}
Min\quad & \{g_{ir}, b_r, i = 1,\dots,m, \quad r = 1,\dots,s\}\\
S.t.\quad & \sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) \le g_{ir} \quad i = 1,\dots,m, \quad r = 1,\dots,s\\
& \sum_{j=1}^{n} \mu_j \left(\frac{1}{y_{rj}}\right) = b_r, \quad r = 1,\dots,s,\\
& \sum_{j=1}^{n} \mu_j = 1,\ \ \mu_j \ge 0,\ \ j = 1,\dots,n.
\end{aligned} \tag{4.18}
$$

Model (4.18) is an input-oriented multi-objective programming problem in input-oriented BCC model in DEA-R.

Model (4.17) and (4.18) can be used to find the suitable target in technology VRS. Therefore, finding the target of the proposed model is not in line with this article because it requires a detailed discussion of other technologies and their comparison. References Mozaffari et al. [11] and Song et al. [53] can be used for this purpose.

5. Numerical example

In this section, we illustrate the algorithm proposed with an example. Consider four DMUs with two inputs and one output as presented in Table 1.

According to the first and second steps of the interactive Z–W method, the input-oriented model (4.15) is first formed using the arbitrary weights $\aleph_1 = 1$ and $\aleph_2 = 0$:

$$
\begin{aligned}
Min\quad & \{(-1)\, g_{11} - (0)\, g_{21}\}\\
s.t.\quad & \frac{2}{4}\mu_1 + \frac{2}{4}\mu_2 + \frac{8}{2}\mu_3 + \frac{2}{5}\mu_4 + H_1 = g_{11}\\
& \frac{3}{4}\mu_1 + \frac{5}{4}\mu_2 + \frac{5}{2}\mu_3 + \frac{4}{5}\mu_4 + H_2 = g_{21}\\
& \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1,\\
& \mu_j \ge 0,\ \ j = 1,2,3,4, \quad H_1, H_2 \ge 0.
\end{aligned}
$$

Step 3. Solving the model leads to the basic solutions $\mu_4 = 1, g_{11} = 0.8$ and $g_{12} = 0.4$ and the non-basic solutions $\mu_3 = 0, \mu_2 = 0, \mu_1 = 0$, $H_1 = 0$ and $H_2 = 0$.

Step 4. To solve the linear programming problem in model (4.15) we obtain the optimal solutions using the General Algebraic Modeling System (GAMS) program as follows:

```
Obj..          Teta=E= mu ('1');

Con1 (i,r)..   Sum(j,(x(j,i)/y(j,r))*mu(j))- g(i,r) + h(i,r)=e= 0;

Con3..      sum(j, mu(j)) =e=1;
```

Table 2
Benchmarking with Z–W Method in DEA-R.

Step	Benchmark in DEA model (3.1)	Benchmark in DEA-R model (4.15)
Step 1	DMU_1 (3.1)	DMU_4 (3.4)
Step 2	DMU_4	DMU_1

So, the solutions for μ_1, μ_2 and μ_3 are $\left(g_{11} = 0.5, g_{12} = 0.75, \mu_1 = 1\right)$, $\left(g_{11} = 0.5, g_{12} = 1.25, \mu_2 = 1\right)$ and $\left(g_{11} = 4, g_{12} = 2.5, \mu_3 = 1\right)$, respectively; moreover, vector ω^j is obtained as follows:

$\omega^1 = (0.1, -0.05)$, $\omega^2 = (0.1, 0.45)$, $\omega^3 = (3.6, 1.7)$.

Step 4. Therefore, μ_2 and μ_3 are not effective and thus we solve the following problem for the μ_1 variable:

$$\begin{aligned} Min \quad & 0.1\gamma_1 - 0.5\gamma_2 \\ s.t. \quad & \gamma_1 + \gamma_2 = 1 \end{aligned}$$

Step 5. The solutions obtained are $\gamma_1 = 0$ and $\gamma_2 = 1$ and the optimal weights are thereby achieved.

Table 2 shows that if we solve the problem using the weights obtained by the Z–W method, the CCR model, DMU_1 is found as a benchmark, while in step 1, the DMU_4 was designated as our benchmark.

6. Case study

In this section, we consider 11 Iranian clothing companies, each using three inputs to produce three outputs. There are two scenarios in relation to input and output data: (1) input and output data are available, in which case models (3.1), (3.5), (3.7), and (4.15) will evaluate the units; and (2) only a ratio of inputs and outputs are available, which prompts us to use model (4.15) for benchmarking.

In all eleven companies, inputs included current debt, total debt in 2014, and current expenses in the second quarter of 2014. Outputs included assets related to the second quarter of 2014, total assets during the year, and benefits obtained in 2014. During the process of evaluation, all eleven companies participating in the first stage refused to disclose their input and output data and provided only the following ratios:

$$\frac{O_1}{I_1}, \frac{O_1}{I_2}, \frac{O_1}{I_3}, \frac{O_2}{I_1}, \frac{O_2}{I_2}, \frac{O_2}{I_3}, \frac{O_3}{I_1}, \frac{O_3}{I_2}, \frac{O_3}{I_3},$$

The ratios are defined in the following manner. The $\frac{O_1}{I_1}$ ratio represents the current ratio, $\frac{O_2}{I_1}$ indicates the quick ratio, and $\frac{O_2}{I_2}$ is defined as the assets ratio. Other ratios are also defined and clear. Meanwhile, in the second stage, the government collected the information regarding the companies' input and output data by enforcement of respective laws. Table 3 summarizes the input and output data of the eleven companies.

Table 3 shows that current liabilities, total debt and current expenses are considered as input. In DEA, DEA-R models, the selection of inputs and outputs is done in a way that we can reduce the inputs and increase the outputs. They are larger than the original units. Inputs are usually factors whose level of reduction is desirable for the decision maker, and this increases the efficiency of the unit under evaluation.

We can provide a similar interpretation for the outputs.

In models in the input, oriented by reducing the input components in the numerator and increasing the level of output components in the denominator, we can reduce the fraction used in the model. We consider components as input that their reduction is desirable for the decision maker and we can increase the efficiency of the unit under evaluation.

We consider outputs as current assets, total assets and previous profit. In the proposed models, because the goal is to increase output, so increasing assets and profits can be important. On the other hand, according to the appraisal manager of clothing companies, it is based on the liquidity ratio, which defines the ratio of assets to debt.

In the proposed model (4.8) which is based on output to input ratios (liquidity ratio). The goal is to increase the ratio $\frac{x_{ij}}{y_{rj}}, i = 1, \ldots, m, r = 1, 1, \ldots, s$. In this regard, we seek to increase outputs (assets) and decrease inputs (costs and liabilities). Therefore, in general, the selection of inputs and outputs with the aim of reducing and increasing them in order to increase the liquidity ratios of companies.

It is obvious that traditional DEA models cannot obtain the benchmarks in the first stage; however, in the case of ratio data, i.e., when data are presented as the input–output ratio $\frac{O}{I}$, the proposed models can specify the benchmarks with the help of the interactive method.

Considering the arbitrary weights $\gamma^1 = \{1, 1, 1, 0, 1, 0, 1, 0, 1\}$, model (4.10) is as follows in the GAMS program:

```
Obj           z= sum (i, sum (r, gamma (i, r) *f (i, r)))
Con1 (i,r).   Sum (j, (y(j,r)/x(j,i))*Lambda(j))- f(i,r) - h(i,r)=e=0
Con3.         Sum(j, lambda(j)) =e=1
```

Table 3
Input and output data of the 11 companies.

DMU	Current Debt	Total Debt	Current Expenses	Current Assets	Total Assets	Earnings Before Interest
	I_1	I_2	I_3	O_1	O_2	O_3
1	1245	7500	89	205	350000	97
2	1381	6300	93	210	810000	89
3	1582	4200	98	151	510000	35
4	1283	9700	95	162	210000	44
5	1951	2100	41	137	360000	13
6	1764	3100	42	146	210000	19
7	2010	2500	28	151	420000	81
8	1725	7650	86	185	840000	35
9	1836	8950	3	225	950000	97
10	1912	1200	72	230	870000	94
11	1121	1382	51	200	850000	90

Table 4
Optimal solutions obtained by the 3rd step of the Z–W Method.

F_{11}	0.18
F_{12}	758.25
F_{13}	0.08
F_{21}	0.14
F_{22}	615.05
F_{23}	0.07
F_{31}	3.92
F_{32}	1.67E+04
F_{33}	1.76

Table 5
4th step of the Z–W Method.

	Max λ_1	Max λ_2	Max λ_3	Max λ_4	Max λ_5	Max λ_6	Max λ_7	Max λ_8	Max λ_9	Max λ_{10}
F_{11}	0.16	0.15	0.1	0.13	0.07	0.08	0.08	0.11	0.12	0.12
F_{12}	281.12	586.53	322.38	163.68	184.52	119.05	208.96	486.96	517.43	455.02
F_{13}	0.08	0.06	0.02	0.03	0.01	0.01	0.04	0.02	0.05	0.05
F_{21}	0.03	0.03	0.04	0.02	0.07	0.05	0.06	0.02	0.03	0.19
F_{22}	46.67	128.57	121.43	21.65	171.43	67.74	168	109.8	106.15	725
F_{23}	0.01	0.01	0.01	0	0.01	0.01	0.03	0	0.01	0.08
F_{31}	2.3	2.26	1.54	1.71	3.34	3.48	5.39	2.15	6.62	3.19
F_{32}	3932.58	8709.68	5204.08	2210.53	8780.49	5000	1.50E+04	9767.44	2.79E+04	1.21E+04
F_{33}	1.09	0.96	0.36	0.46	0.32	0.45	2.89	0.41	2.85	1.31

Table 6
5th step of the Z–W Method.

ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}
0.02	0.03	0.08	0.05	0.11	0.1	0.1	0.07	0.06	0.06
477.13	171.72	435.87	594.57	573.73	639.2	549.29	271.29	240.82	303.23
0	0.02	0.06	0.05	0.07	0.07	0.04	0.06	0.03	0.03
0.11	0.11	0.1	0.12	0.07	0.09	0.08	0.12	0.11	−0.05
568.38	486.48	493.62	593.4	443.62	547.31	447.05	505.25	508.9	−109.95
0.06	0.06	0.06	0.07	0.06	0.06	0.04	0.07	0.06	−0.01
1.62	1.66	2.38	2.21	0.58	0.44	−1.47	1.77	−2.7	0.73
12767.42	7990.32	11495.92	14489.47	7919.51	11700	1700	6932.56	−11200	4600
0.67	0.8	1.4	1.3	1.44	1.31	−1.13	1.35	−1.09	0.45

According to the third step of the interactive Z–W Method, the solutions can be obtained as in Table 4.

Step 4. For non-basic variables, we solve model (4.8). The solutions for non-basic variables λ_l and $l = 1, \dots, 10$ are obtained as in Table 5.

Step 5. The value of ω^1 can be calculated using Eq. (4.9) and the optimal solutions provided in Table 6. For instance, for ω^1 we have:

$\omega^1 = \left(\omega_{11}, \omega_{21}, \dots, \omega_{91}\right) = (\frac{0.18-0.16}{1}, \frac{758.25-281.12}{1}, \dots, \frac{1.76-1.09}{1}) = (0.02,\ 477.13,\ \dots, 0.67)$

Related data are presented in the first column of Table 6. The rest of ω^1 values are obtained in the same manner (see Table 6).

Step 6. As can be seen, all non-basic variables except λ_7, λ_9, and λ_{10} are ineffective because all corresponding ω_{kl} values are positive. Thus, we solve the suggested models for these 3 variables. The following table is drawn for ω^7, ω^9 and ω^{10} (see Table 7).

Table 7
6th step of the Z–W Method.

	ω^7	ω^9	ω^{10}
Objective function coefficients	−0.172	−11200	−109.95
γ_1	0	0	0
γ_2	0	0	0
γ_3	0	0	0
γ_4	0	0	0
γ_5	0.002	0	1
γ_6	0	0	0
γ_7	0	0	0
γ_8	0	1	0
γ_9	0.998	0	0

In Table 8, a comparison of CCR efficiency scores and benchmarks in DEA and DEA-R models is shown. Therefore, DMU11 may be a target for most DMUs.

Table 8
CCR efficiency scores and benchmarks in DEA and DEA-R models.

DMU	CCR efficiency in DEA model (3.1)	CCR efficiency in DEA-R model (3.4)	Benchmark in DEA-R model (4.10)	Benchmark in DEA-R model (4.15)
DMU_1	0.9704	0.9691	DMU_{11}	DMU_{11}
DMU_2	0.8523	0.8518	DMU_{11}	DMU_{11}
DMU_3	0.5350	0.5348	DMU_{11}	DMU_{11}
DMU_4	0.7177	0.7070	DMU_{11}	DMU_{11}
DMU_5	0.7438	0.6560	DMU_{11}, DMU_7	DMU_9, DMU_{10}
DMU_6	0.7386	0.6260	DMU_{11}, DMU_7, DMU_9	DMU_{11}, DMU_{10}, DMU_9
DMU_7	1.0	1.0	DMU_7	DMU_7
DMU8	0.6422	0.6421	DMU_{11}	DMU_{11}
DMU_9	1.0	1.0	DMU_9	DMU9
DMU_{10}	1.0	1.0	DMU_{10}	DMU_{10}
DMU_{11}	1.0	1.0	DMU_{11}	DMU_{11}

Step 7. If we assume that the manager's response to exchange ω^9 is "Yes" and his/her response to ω^{10} and ω^7 is "No", then the following model is solved.

$$Max\ 0.002\ \gamma_5 + 0.998\ \gamma_9$$

$$\begin{aligned}
&s.t\ 0.002\ \gamma_5 + 0.998\ \gamma_9 \geq \varepsilon \\
&\gamma_5 \geq \varepsilon \\
&\gamma_6 \leq -\varepsilon \\
&\gamma_9 = 1 + \gamma_8 + \gamma_7 + \gamma_6 + \gamma_5 + \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1
\end{aligned}$$

Step 8. Therefore, $\gamma_{11} = 0.5$, $\gamma_{22} = 0.5$ and the rest of γ_{ir} values equal to zero. Furthermore, considering the objective function's value, the optimal weight is equal to 0.001. Moreover, if the manager's response is "Yes" to exchanging ω^{10} and "No" for ω^7 and ω^9, the optimal weights of γ_{11} and γ_{32} would be 0.998 and 0.001, respectively. Now, based on the value of the objective function (0.001), the rest of γ_{ir} values will equal to zero.

The first column shows that CCR efficiency scores in DEA models are greater than or equal to their corresponding scores in DEA-R models (second column). Benchmarks were obtained for all DMUs using DEA and DEA-R models in the GAMS program, as presented in the third and fourth columns, respectively. For instance, DMU_{11} is a benchmark for decision-making units {1, 2, 3, 4, 6, and 11} in both DEA and DEA-R with this unit once again being selected as a benchmark in the first step using the Z–W method. Finally, in the seventh step, where the manager's first answer (Yes) was taken into account and model (4.11a) and (4.11b) was solved, DMU_{10} is specified as the benchmark. However, if we apply the second assumption for manager's response (No) to model (4.11), we arrive at DMU_9 as our benchmark (See Table 9).

However, in cases that we have both ratio and volume data simultaneously, the convexity constraint needs to be revised. Therefore, in this paper, we attempted to evaluate the DMUs via the interactive Z–W method (see Table 9).

7. Conclusion

Generally, in DEA and DEA-R models, when there are several inputs and one output and/or vice versa, the scale efficiencies are equal to each other. Also, there are no weight restrictions due to the use of the non-Archimedean number ε and because of this, pseudo-inefficiency might occur in DEA. DEA models based on Ratio Analysis are generally appropriate for ratio data. Moreover, several features of DEA are similar to DEA-R. When solving multi-objective linear programming problems via interactive methods, there is no need to gather any information from the decision-makers (DMs) prior to model solving. The required information is collected during the solving process. The Z–W method is an interactive multi-objective linear programming approach that can be used to obtain proper benchmarks for inefficient units on the efficient frontier. In this paper we used the Z–W method to obtain

Table 9
Benchmarking based on the Z–W Method.

Algorithm Step	Step 1	Step 2	Step 3
Benchmark in DEA-R	DMU_{11}	DMU_{10}	DMU_9

different benchmarks based on DEA-R models rather than DEA models. It was demonstrated that by increasing the number of constraints in the DEA-R model, certain computational problems arose in the interactive method. For future research, we can use other methods of MOLP problem solving with the objective to decrease the amount of computations. We also suggest determining the Malmquist Productivity Index in DEA-R and making a comparison of optimized weights and their relationships in the production possibility set.

References

[1] Farrell MJ. The measurement of productive efficiency. J R Stat 1957;120(3):253–81.
[2] Charnes A, Cooper WW, Rhodes E. Measuring the efficiency of decision-making units. European J Oper Res 1978;2(6):429–44.
[3] Banker RD, Charnes A, Cooper WW. Some models for estimating technical and scale inefficiency in data envelopment analysis. Manag Sci 1984;30(9):1078–92.
[4] Despic O, Despic M, Paradi JC. DEA-R: ratio-based comparative efficiency model, its mathematical relation to DEA and its use in applications. J Product Anal 2007;28(1–2):33–44.
[5] Zionts S, Wallenius J. An interactive programming method for solving the multiple criteria problem. Manag Sci 1976;22(6):629–725.
[6] Emrouznejad A, GhR Amin. DEA models for ratio data: Convexity consideration. Appl Math Model 2009;33(1):486–98.
[7] Olesen OB, Petersen NC, Podinovski VV. Efficiency analysis with ratio measures. European J Oper Res 2015;245(2):446–62.
[8] Olesen OB, Petersen NC, Podinovski VV. Efficiency measures and computational approaches for envelopment analysis models with ratio inputs and outputs. European J Oper Res 2017;261(2):640–55.
[9] Fernandez-Castro A, Smith P. Towards a general non-parametric model of corporate performance. Omega 1994;22(3):237–49.
[10] Liu WB, Zhang DQ, Meng W, Li XX, Xu F. A study of DEA models without explicit inputs. Omega 2011;39(5):472–80.
[11] Mozaffari MR, Dadkhah F, Jablonsky J, Wanke PF. Finding efficient surfaces in DEA-R models. Appl Math Comput 2020;386:125497.
[12] Allen R, Athanassopoulis A, Dyson RG, Thanassoulis E. Weights restrictions and value judgements in data envelopment analysis: evolution, development and future directions. Ann Oper Res 1997;73:13–34.
[13] Golany B. An interactive MOLP procedure for the extension of DEA to effectiveness analysis. J Oper Res Soc 1988;39(8):725–34.
[14] Athanassopoulis A. Decision support for target-based resource allocation of public services in multiunit and multilevel systems. Manag Sci 1998;39(10):1261–4.
[15] Dyson RG, Thannassoulis E. Reducing weight flexibility in data envelopment analysis. J Oper Res Soc 1988;39(6):563–76.
[16] Thompson RG, Dharmapala PS, Thrall RM. Linked-cone DEA profit ratios and technical efficiency with application to illinois coal mines. Int J Prod Econ 1995;39(1–3):99–115.
[17] Wong YHB, Beasley JE. Restricting weight flexibility in data envelopment analysis. J Oper Res Soc 1990;41(9):829–35.
[18] Charnes A, Cooper WW. Polyhedral cone-ratio DEA models with an illustrative application to large commercial banking. J Econom 1990;46:73–9.
[19] Charnes A, Cooper WW, Huang ZM, Sun DB. Polyhedral cone ratio DEA models with an illustrative application to large commercial banks. J Econom 1990;30:91–107.
[20] Zhu J. Data envelopment analysis with preference structure. J Oper Res Soc 1996;47(1):136–50.
[21] Joro R, Korhonen P, Wallenius J. Structural comparison of data envelopment analysis and multiple objective linear programming. Manag Sci 1998;44(7):962–70.
[22] Joro R, Korhonen P, Zionts S. An interactive approach to improve estimates of value efficiency in data envelopment analysis. European J Oper Res 2003;149:688–99.
[23] Wong BYH, Luque M, Yang JB. Using interactive multiobjective methods to solve DEA problems with value judgements. Comput Oper Res 2009;36:623–36.
[24] Yang JB, Wong BYH, Xu DL, Stewart TJ. Integrating DEA-oriented performance assessment and target setting using interactive MOLP methods. European J Oper Res 2009;195(1):205–22.
[25] Hosseinzadeh Lotfi F, Jahanshaloo GR, Soltanifar M, Ebrahimnejad A, Manosourzadeh SM. Relationship between MOLP and DEA based on output-orientated CCR dual model. Expert Syst Appl 2010;37(6):4331–6.
[26] Hosseinzadeh Lotfi F, Jahanshaloo GR, Soltanifar M, Ebrahimnejad A, Manosourzadeh SM. Relationship between MOLP and DEA based on output-orientated CCR dual model. Expert Syst Appl 2010;37(6):4331–6.
[27] Zohrehbandian M. Using zionts–wallenius method to improve estimate of value efficiency in DEA. Appl Math Model 2011;35:3769–76.
[28] Ebrahimnejad A, Hosseinzadeh Lotfi F. Equivalence relationship between the general combined-oriented CCR model and the weighted minimax MOLP formulation. J King Saud Univ –Sci 2012;24:47–54.
[29] Ebrahimnejad A, Tavana M. An interactive MOLP method for identifying target units in output-oriented DEA models: The NATO enlargement problem. Measurement 2014;52:124.
[30] Ebrahimnejad A, Tavana M, Hosseinzadeh Lotfi F, Shahverdi R, Yousefpour M. A three-stage data envelopment analysis model with application to banking industry. Measurement 2014;49:308–19.
[31] Kao HY, Chan ChY, Wu DJ. A multi-objective programming method for solving network DEA. Appl Soft Comput 2014;24:406–13.
[32] Gerami J. An interactive procedure to improve estimate of value efficiency in DEA. Expert Syst Appl 2019;137:29–45.
[33] Soltani N, Lozano S. Interactive multiobjective DEA target setting using lexicographic DDF. RAIRO Oper Res 2020;54(6):1703–22.
[34] Cook WD, Ruiz JL, Sirvent I, Zhu J. Within-group common benchmarking using dea. European J Oper Res 2017;256(3):901–10.
[35] Lim S, Bae H, Lee LH. A study on the selection of benchmarking paths in DEA. Expert Syst Appl 2011;38(6):7665–73.
[36] Lozano S, Calzada-Infante L. Computing gradient-based stepwise benchmarking paths. Omega 2018;81:195–207.
[37] Cook WD, Tone K, Zhu J. Data envelopment analysis: prior to choosing a model. Omega 2014;44:1–4.
[38] Bogetoft P. Performance benchmarking: measuring and managing performance. Springer Science & Business Media; 2013.
[39] Lozano S, Soltani N. DEA target setting using lexicographic and endogenous directional distance function approaches. J Product Anal 2021;50(1–2):55–70.
[40] An Q, Meng F, Xiong B, Wang Z, Chen X. Assessing the relative efficiency of Chinese high-tech industries: a dynamic network data envelopment analysis approach. Ann Oper Res 2018;1–23.
[41] Ruiz LJ, Sirvent I. Common benchmarking and ranking of units with DEA. Omega 2016;65:1–9.
[42] Wei CK, Chen LC, Li RK, Tsai CH. A study of developing an input-oriented ratio-based comparative efficiency model. Expert Syst Appl 2011;38(3):2473–7.

[43] Wei CK, Chen LC, Li RK, Tsai CH. Exploration of efficiency underestimation of CCR model. Expert Syst Appl 2011;38(4):3155–60.
[44] Wei CK, Chen LC, Li RK, Tsai CH. Using the DEA-r model in the hospital industry to study the pseudo-inefficiency problem. Expert Syst Appl 2011;38(3):2172–6.
[45] Mozaffari MR, Kamyab P, Gerami J, Jablonsky J. Cost and revenue efficiency in DEA-r models. Comput Ind Eng 2014b;78:188–94.
[46] Hatami-Marbini A, Toloo M. Data envelopment analysis models with ratio data: A revisit. Comput Ind Eng 2021;133:331–8.
[47] Gerami J, Mozaffari MR, Wanke PF. A multi-criteria ratio-based approach for two-stage data envelopment analysis. Expert Syst Appl 2020;158:113508.
[48] Gerami J, Kiani Mavi R, Farzipoor Saen R, Kiani Mavi N. A novel network DEA-r model for evaluating hospital services supply chain performance. Ann Oper Res 2020. http://dx.doi.org/10.1007/s10479-020-03755-w.
[49] Chen Y, Ali AI. Output-input ratio analysis and DEA frontier. European J Oper Res 2002;142(3):476–9.
[50] Meng W, Zhang DQ, Liu PH, Peng ZL, Zhang WY. Analysis of a questionnaire about the CAS basic research intellectual property rights. Bull Chin Academ Sci 2005;30:312–20.
[51] Mahlberg B, Obersteiner M. Remeasuring the HDI by data envelopment analysis. 2001, Report IR-01-069, http://pure.iiasa.ac.at/id/eprint/6455/1/IR-01-069.pdf.
[52] Steuer RE, Choo EU. An interactive weighted Tchebycheff procedure for multiple objective programming. Math Program 1983;26(3):326–44.
[53] Song M, Wu J, Wang Y. An extended aggregated ratio analysis in DEA. J Syst Sci Syst Eng 2011;20(2):249–56.